Some lower bounds for the L-intersection number of graphs

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Abstract

For a set of non-negative integers L, the L-intersection number of a graph is the smallest number l for which there is an assignment of subsets $A_v \subseteq \{1, \dots, l\}$ to vertices v, such that every two vertices u, v are adjacent if and only if $|A_u \cap A_v| \in L$. The bipartite L-intersection number is defined similarly when the conditions are considered only for the vertices in different parts. In this paper, some lower bounds for the (bipartite) L-intersection number of a graph for various types L in terms of the minimum rank of graph are obtained. To achieve the main results we employ the inclusion matrices of set systems and show that how the linear algebra techniques give elegant proof and stronger results in some cases.

Keywords: Set intersection representation; L-Intersection number; Bipartite set intersection representation; Bipartite L-intersection number.

1 Introduction

A *graph representation* is an assignment on the vertices of graph to a family of objects satisfying certain conditions and a rule that determines from the objects whether or not two vertices are adjacent. In the literature, different types of graph representations such as the set intersection representation [5, 8] and the vector representation [10, 11, 12] are studied.

A basic graph representation is the *set intersection representation* in which an assignment of sets to vertices determines an edge between two vertices if the intersection of the corresponding sets satisfies a certain given rule. Precisely, let G be a finite simple graph with vertex set V and L be a subset of non-negative integers. An *L-intersection representation* of G, assign to every vertex $v \in V$ a finite set A_v, such that two vertices u and v are adjacent if and only if $|A_u \cap A_v| \in L$. We are interested in the minimum size of the universe of the sets, $| \cup_{v \in V} A_v|$. This parameter is denoted by $\Theta_L(G)$ and called the *L-intersection number* of G [5].

For bipartite graph G with a fixed vertex partition $V = V_1 \cup V_2$, the definition can be modified by relaxing the condition inside the partition sets (since for vertices inside a partite set, we know they are not adjacent). Indeed, a *bipartite L-intersection representation* of graph G, for a given set $L \subseteq \{0, 1, 2, \dots\}$, assign to every vertex $v \in V$ a finite set A_v, such that two vertices u, v from different partite sets are adjacent if and only if $|A_u \cap A_v| \in L$. The relaxed measure of the L-intersection number is denoted by $\theta_L(G)$ [8]. It is clear that $\Theta_L(G) \geq \theta_L(G)$ for every bipartite graph G and set L.

One of the important measures regarding set intersection representations is finding the optimal representation for a graph by considering different sets L. Indeed, the *absolute dimension* of G is defined as $\Theta(G) = \min_L \Theta_L(G)$ over all sets L of non-negative integers (similarly, *the bipartite absolute dimension* is $\theta(G) = \min_L \theta_L(G)$). This concept has close connection to the log-rank conjecture in communication complexity [8]. Also, finding explicit lower bounds for absolute dimension has important consequence in the complexity theory [8, 14, 15]. However, an easy counting argument, shows that there exist graphs of order n with absolute dimension $\Omega(n)$ [8].

A twin-free graph is a graph without any pair of vertices with $N(u) - \{v\} = N(v) - \{u\}$, where $N(x)$ is the set of vertices adjacent to x. As a matter of fact, for every twin-free graph G of order n, $\Theta(G) \geq \log_2 n$. This lower bound is obtained from the fact that in such a graph no pair of vertices could be assigned the same set in an optimal representation. Although, this lower bound is obtained simply, the question of finding an explicit construction for graph G such that $\theta(G) = \Omega(\log n)$ or even $\Theta(G) = \Omega(\log n)$, is going to be a very challenging problem [1, 8]. It is easy to see that if H is a maximal twin-free induced subgraph of G, then $\Theta_L(H) \leq \Theta_L(G)$ and $\theta_L(H) = \theta_L(G)$, for every set L. Thus, every lower bound for the L-intersection number of H is a lower bound for the L-intersection number of G. Throughout this paper we consider twin-free graphs with no isolated vertex.

A good summary on the known results on the L-intersection number is given by Jukna [8] (for more results in this subject see [2, 3, 4, 7]). The most studied problems in this concept are related to the threshold type $L = \{1, 2, \dots\}$ which in the general case is known as the edge clique covering number, denoted by $\Theta_1(G)$.

The complement of a graph G is denoted by $\overline{G}$. Also, by a bipartite complement of a bipartite graph $G = (V_1 \cup V_2, E)$ we mean the bipartite graph $G^c = (V_1 \cup V_2, E^c)$, where $E^c = (V_1 \times V_2) \backslash E$.

Theorem 1.1. [3] *Let $L = \{0, 1, \dots, k-1\}$ for some integer k. Then, for every graph G, $\Theta_L(\overline{G}) \geq (\Theta_1(G))^{1/k}$.*

The bipartite L-intersection number for $L = \{1, 2, \dots\}$ corresponds to the well-known parameter, the edge biclique covering number [9]. The bipartite L-intersection number for various sets L are studied in [8]. Specially the following lower bounds are obtained when $L = \{l : l \pmod{p} \in R\}$ for a given subset R of residues module p, such set L is called *modular type*.

Theorem 1.2. [8] *Let p be a prime number and R be a subset of residues module p with $|R| = r$. If $L = \{l : l \pmod{p} \in R\}$, then for every graph G of order n and maximum degree Δ,*

(i) $\theta_L(G^c) \geq (n/\Delta)^{\frac{1}{r}}$.

(ii) $\theta_L(G) \geq (\frac{1}{r} n/\Delta)^{\frac{1}{p-1}}$.

In this paper we are concerned with finding lower bounds for (bipartite) L-intersection number of graphs for various types L. To do this, our main tools are linear algebra techniques via inclusion matrices. We show how these techniques give elegant proofs and stronger results in some cases.

The structure of the paper is as follows. First, in Section 2, we present basic technique which we have used through the paper. Then, in Section 3, we obtain some lower bounds for the L-intersection number for modular types and finite sets L. By the similar method, in Section 4, we find some lower bounds for the bipartite L-intersection number which improve the bounds in Theorem 1.2. Finally, in Section 5, we consider the uniform intersection set representation of graphs, where all sets assigned to the vertices have the same size, and obtain some lower bounds for such L-intersection number.

2 The Key Technique

This section deals with the basic tools which are used to get the main results of the paper.

We start with the definition of the rank of a graph.

Let $\mathcal{M}_n(\mathbb{F})$ be the set of all $n \times n$ matrices over a field $\mathbb{F}$ and $\mathcal{S}_n(\mathbb{F})$ be the subset of all symmetric matrices in $\mathcal{M}_n(\mathbb{F})$. We consider $\mathcal{S}_n(\mathbb{F})$ and $\mathcal{M}_n(\mathbb{F})$ as vector spaces over the field $\mathbb{R}$. For $A \in \mathcal{S}_n(\mathbb{F})$, the graph of A, denoted by $\mathcal{G}(A)$, is a graph with vertex set $\{1, \ldots, n\}$ and edge set $\{ij : A_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the entries of the diagonal of A are ignored in determining $\mathcal{G}(\mathcal{A})$.

The *minimum rank* [13] of a graph G over a field $\mathbb{F}$ is defined to be

$$\mathrm{mr}_{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}_n(\mathbb{F}),\ \mathcal{G}(A) \cong G\},$$

where $\cong$ means the graph isomorphism relation.

In the case of bipartite graph, for convenience we consider the bipartite adjacency matrix. The bipartite adjacency matrix of an $n \times n$ bipartite graph G with a vertex partition $V = V_1 \cup V_2$, denoted by $A_b(G)$, is a $(0, 1)$-matrix whose rows correspond to the vertices of V_1 and its columns correspond to the vertices of V_2, and the (i, j) entry of $A_b(G)$ is 1 if and only if vertex i is adjacent to vertex j. For $A \in \mathcal{M}_n(\mathbb{F})$, the bipartite graph $\mathcal{G}_b(A)$ is a graph with bipartite set V_1 and V_2 corresponding to the rows and the columns of A, respectively, and edges $\{ij : A_{ij} \neq 0\}$.

The *bipartite minimum rank* of a bipartite graph G over a field $\mathbb{F}$ is defined to be

$$\mathrm{bmr}_{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{M}_n(\mathbb{F}),\ \mathcal{G}_b(A) \cong G\}.$$

It can be easily seen that, for every bipartite graph G, $\mathrm{mr}_{\mathbb{F}}(G) = 2\mathrm{bmr}_{\mathbb{F}}(G)$. For convenience, when $\mathbb{F} = \mathbb{R}$, we denote $\mathrm{mr}_{\mathbb{F}}(G)$ and $\mathrm{bmr}_{\mathbb{F}}(G)$ by $\mathrm{mr}(G)$ and $\mathrm{bmr}(G)$, also for $\mathbb{F} = \mathbb{Z}_p$ we denote them by $\mathrm{mr_p}(G)$ and $\mathrm{bmr_p}(G)$, respectively.

Let $\mathcal{F}$ and $\mathcal{T}$ be two families of subsets of set $[l] = \{1, \ldots, l\}$. The $(\mathcal{F}, \mathcal{T})$-*inclusion matrix*, denoted by $I_l(\mathcal{F}, \mathcal{T})$ is a $(0, 1)$-matrix whose rows and columns are labelled by the members of $\mathcal{F}$ and $\mathcal{T}$, respectively. The (F, T) entry of $I_l(\mathcal{F}, \mathcal{T})$ will be 1 or 0 according to whether or not $T \subseteq F$. In the case that $\mathcal{T}$ is the family of all t-subsets of $[l]$, we denote the matrix by $I_l(\mathcal{F}, t)$ and call it the *t-inclusion matrix* of $\mathcal{F}$. When $\mathcal{F}$ is the family of all i-subsets of $[l]$, the corresponding t-inclusion matrix is denoted by $I_l(i, t)$. Let $A_t(\mathcal{F}, \mathcal{T}) = I_l(\mathcal{F}, t) I_l(\mathcal{T}, t)^T$, we call $A_t(\mathcal{F}, \mathcal{T})$ the *t-intersection matrix* of $\mathcal{F}$ and $\mathcal{T}$ [6]. Indeed, $A_t(\mathcal{F}, \mathcal{T})$ is an $|\mathcal{F}| \times |\mathcal{T}|$ matrix where its (F, T) entry is $\binom{|F \cap T|}{t}$. Moreover,

$$\operatorname{rank}(A_t(\mathcal{F}, \mathcal{T})) \leq \operatorname{rank}(I_l(\mathcal{F}, t)) \leq \binom{l}{t}. \tag{1}$$

Proposition 2.1. *Let $\mathcal{F}$ and $\mathcal{T}$ be two families of subsets of $[l]$. If $M = \sum_{t=0}^{k} c_t A_t(\mathcal{F}, \mathcal{T})$, where for any $0 \leq t \leq k$, c_t is a real number, then* $\operatorname{rank}(M) \leq \sum_{t=0}^{k} \binom{l}{t}$.

Proof. By the definition of M, the row space of M is a subspace of the vector space spanned by the rows of $A_t(\mathcal{F}, \mathcal{T})$, $0 \leq t \leq k$, and thus by the subadditivity of the rank function and relation (1),

$$\operatorname{rank}(M) \leq \sum_{t=0}^{k} \operatorname{rank}(A_t(\mathcal{F}, \mathcal{T})) \leq \sum_{t=0}^{k} \binom{l}{t}.$$

□

Recall that for every non-negative integer t the binomial coefficient $\binom{x}{t}$ over a field $\mathbb{F}$ is defined as

$$\binom{x}{t} = \frac{1}{t!} x(x-1)\dots(x-t+1).$$

Note that the polynomials $\binom{x}{0}, \binom{x}{1}, \dots, \binom{x}{d}$ form a basis for space of polynomials of degree at most d. Thus, we have the following well-known proposition.

Proposition 2.2. *Every polynomial $f(x)$ of degree $d \geq 0$ can uniquely be expressed as the linear combination of the binomial coefficients, $\binom{x}{i}$, $0 \leq i \leq d$.*

3 Lower bounds for the L-intersection number

In this section, we present some lower bounds for the L-intersection number of a graph G for modular types and finite sets L in terms of the minimum rank of G.

Theorem 3.1. *If p is a prime number, $R \subset \mathbb{Z}_p$ with $|R| = r$, $L = \{l : l \pmod{p} \in R\}$ and $\mathcal{F}$ is an L-intersection representation of a graph G, then we have the following statements, where the matrices are considered over $\mathbb{Z}_p$.*

(i) *There are real numbers c_t, $0 \leq t \leq r$, such that if $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{F})$, then $\mathcal{G}(M) \cong \overline{G}$.*

(ii) *There are real numbers c_t, $0 \leq t \leq p-1$, such that if $M = \sum_{t=0}^{p-1} c_t A_t(\mathcal{F}, \mathcal{F})$, then $\mathcal{G}(M) \cong G$.*

Proof. Let $\mathcal{F} = \{F_1, \dots, F_n\}$, where for any $1 \le i \le r$, F_i is the set assigned to the vertex i, and $R = \{\rho_1, \rho_2, \dots, \rho_r\}$. Recall that, $A_t(\mathcal{F}, \mathcal{F})$ is an $n \times n$ matrix, where the (F_u, F_v) entry is $\binom{|F_u \cap F_v|}{t}$.

(i) By Proposition 2.2, there are real numbers c_t, $0 \le t \le r$, such that for every non-negative integer x,

$$\prod_{i=1}^{r}(x - \rho_i) \equiv \sum_{t=0}^{r} c_t \binom{x}{t} \qquad \pmod{p}.$$

Thus, for $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{F})$, the (u, v) entry of M is equal to $\prod_{i=1}^{r}(|F_u \cap F_v| - \rho_i)$ in $\mathbb{Z}_p$. Therefore, M is a symmetric matrix. In view of the definition of L, one can see that for every $u \ne v$, the (u, v) entry is zero in $\mathbb{Z}_p$ if and only if vertex u is adjacent to vertex v in G. Hence, over the $\mathbb{Z}_p$,

$$\mathcal{G}(M) \cong \overline{G}.$$

(ii) By Proposition 2.2, there are real numbers c_t, $0 \le t \le p-1$, such that for every non-negative integer x,

$$\sum_{i=1}^{r}(1 - (x - \rho_i)^{p-1}) \equiv \sum_{t=0}^{p-1} c_t \binom{x}{t} \qquad \pmod{p}.$$

Thus, for $M = \sum_{t=0}^{p-1} c_t A_t(\mathcal{F}, \mathcal{F})$, the (F_u, F_v) entry in M is equal to $\sum_{i=1}^{r}[1 - (|F_u \cap F_v| - \rho_i)^{p-1}]$. Hence, by the Fermat's little theorem, for every two vertices u and v, the (F_u, F_v) entry in M is zero in $\mathbb{Z}_p$ if and only if vertex u is not adjacent to vertex v. Therefore, $\mathcal{G}(M) \cong G$. □

Now by Proposition 2.1 and Theorem 3.1, we are ready to get the main result of this section.

Theorem 3.2. *Let p be a prime number and R be a subset of residues module p with $|R| = r$. If $L = \{l : l \pmod{p} \in R\}$, then for every graph G,*

(i) $\mathrm{mr_p}(\overline{G}) \le \displaystyle\sum_{t=0}^{r} \binom{\Theta_L(G)}{t}$.

(ii) $\mathrm{mr_p}(G) \le \displaystyle\sum_{t=0}^{p-1} \binom{\Theta_L(G)}{t}$.

Proof. Let $\mathcal{F}$ be an optimal L-intersection representation of G.

(i) By Theorem 3.1 there are real numbers c_t, $0 \le t \le r$, such that in $\mathbb{Z}_p$ for $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{F})$, we have $\mathcal{G}(M) \cong \overline{G}$. Thus by Proposition 2.1, $\mathrm{mr_p}(\overline{G}) \le \mathrm{rank}(M) \le \sum_{t=0}^{r} \binom{\Theta_L(G)}{t}$.

(ii) By Theorem 3.1 there are real numbers c_t, $0 \leq t \leq p-1$, such that in $\mathbb{Z}_p$ for $M = \sum_{t=0}^{p-1} c_t A_t(\mathcal{F}, \mathcal{F})$, we have $\mathcal{G}(M) \cong G$. Thus by Proposition 2.1, $\mathrm{mr_p}(G) \leq \mathrm{rank}(M) \leq \sum_{t=0}^{p-1} \binom{\Theta_L(G)}{t}$. $\square$

Using the following approximation for the binomial coefficients, we obtain lower bounds for $\Theta_L(\overline{G})$ and $\Theta_L(G)$ in terms of the minimum rank of G.

It can be seen that, for positive integers x and $s > 1$, we have

$$\sum_{i=0}^{s} \binom{x}{i} \leq x^s. \tag{2}$$

Corollary 3.3. *Let p be a prime number and R be a subset of residues module p with $|R| = r$, where $r > 1$. If $L = \{l : l \pmod p \in R\}$, then for every graph G,*

(i) $\Theta_L(\overline{G}) \geq (\mathrm{mr_p}(G))^{\frac{1}{r}}$.

(ii) $\Theta_L(G) \geq (\mathrm{mr_p}(G))^{\frac{1}{p-1}}$.

Note that the proof of Theorem 3.1(i), works for field $\mathbb{R}$ and any finite set L. Thus, by the similar argument the lower bounds in terms of $\mathrm{mr}(G)$ for $\Theta_L(\overline{G})$ are obtained. Hence, we have the following theorem.

Theorem 3.4. *If L is a finite set of size s, where $s > 1$, then for every graph G, $\Theta_L(\overline{G}) \geq (\mathrm{mr}(G))^{\frac{1}{s}}$.*

4 Lower bounds for the bipartite L-intersection number

This section deals with the bipartite L-intersection number of graphs for modular types and finite sets L. Here, by defining some appropriate inclusion matrices, we obtain lower bounds for $\theta_L(G)$ in terms of the bipartite minimum rank of G.

By the similar argument to Theorem 3.1 next theorem can be proved for the case of bipartite graphs.

Theorem 4.1. *If p is a prime number, $R \subset \mathbb{Z}_p$ with $|R| = r$, $L = \{l : l \pmod p \in R\}$ and $\mathcal{F} \cup \mathcal{T}$ is a bipartite L-intersection representation of a bipartite graph G such that $\mathcal{F}$ and $\mathcal{T}$ corresponds to the different parts of G, then we have the following statements, where the matrices are considered over $\mathbb{Z}_p$.*

(i) *There are real numbers c_t, $0 \leq t \leq r$, such that if $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{T})$, then $\mathcal{G}_b(M) \cong G^c$.*

(ii) *There are real numbers c_t, $0 \leq t \leq r$, such that if $M = \sum_{t=0}^{p-1} c_t A_t(\mathcal{F}, \mathcal{T})$, then $\mathcal{G}_b(M) \cong G$.*

Now by Proposition 2.1 and Theorem 4.1, we are ready to get the main result of this section.

Theorem 4.2. *Let p be a prime number and R be a subset of residues module p with $|R| = r$. If $L = \{l : l \pmod p \in R\}$, then for every bipartite graph G,*

(i) $\mathrm{bmr_p}(G^c) \leq \sum_{t=0}^{r} \binom{\theta_L(G)}{t}$.

(ii) $\mathrm{bmr_p}(G) \leq \sum_{t=0}^{p-1} \binom{\theta_L(G)}{t}$.

Proof. Let $\mathcal{F} \cup \mathcal{T}$ be an optimal bipartite L-intersection representation of G such that $\mathcal{F}$ and $\mathcal{T}$ corresponds to the different parts of G.

(i) By Theorem 4.1 there are real numbers c_t, $0 \leq t \leq r$, such that in $\mathbb{Z}_p$ for $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{T})$, we have $\mathcal{G}_b(M) \cong G^c$. Thus by Proposition 2.1, $\mathrm{bmr_p}(G^c) \leq \sum_{t=0}^{r} \binom{\theta_L(G)}{t}$.

(ii) By Theorem 4.1 there are real numbers c_t, $0 \leq t \leq p-1$, such that in $\mathbb{Z}_p$ for $M = \sum_{t=0}^{p-1} c_t A_t(\mathcal{F}, \mathcal{T})$, we have $\mathcal{G}_b(M) \cong G$. Thus by Proposition 2.1, $\mathrm{bmr_p}(G) \leq \sum_{t=0}^{p-1} \binom{\theta_L(G)}{t}$. □

It is known that if in the above theorem, L is the set of odd numbers, i.e. $p = 2$ and $R = \{1\}$, then for every bipartite graph G, $\theta_L(G) = \mathrm{mr}_{\mathbb{Z}_2}(G)$ [8]. This shows that the above lower bounds are tight.

From Theorem 4.2, by the approximation (2) for the binomial coefficients, we get the following corollary.

Corollary 4.3. *Let p be a prime number and R be a subset of residues module p with $|R| = r$, where $r > 1$. If $L = \{l : l \pmod p \in R\}$, then for every bipartite graph G,*

(i) $\theta_L(G^c) \geq (\mathrm{bmr_p}(G))^{\frac{1}{r}}$.

(ii) $\theta_L(G) \geq (\mathrm{bmr_p}(G))^{\frac{1}{p-1}}$.

By the above lower bounds we obtain an alternative proof of Theorem 1.2 as follows.

A bipartite $n \times n$ graph $G = (V_1 \cup V_2, E)$ is *increasing* if it is possible to enumerate its vertices $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$ so that $x_i y_i \in E$ and $x_i y_j \notin E$ for all $i > j$. Jukna [8] proves that every bipartite $n \times n$ graph G of maximum degree Δ, with no isolated vertices, contains an induced bipartite $(n/\Delta) \times (n/\Delta)$ increasing subgraph.

Clearly, if H is the induced bipartite $(n/\Delta) \times (n/\Delta)$ increasing subgraph of G, then $\mathrm{bmr}_{\mathbb{F}}(G) \geq \mathrm{bmr}_{\mathbb{F}}(H)$. Moreover, the adjacency matrix of H is upper triangular with non-zero diagonal entry. Thus, $\mathrm{bmr}_{\mathbb{F}}(G) \geq n/\Delta$ over any field $\mathbb{F}$. Hence, Corollary 4.3 implies Theorem 1.2.

5 Uniform set intersection representation

In this section, we consider the set intersection representation of graphs that have constraints on the size of sets assigned to the vertices. In fact, if all sets assigned to the vertices are of the same size, say k, then the representation is called the *k-uniform intersection representation.* The *(L, k)-intersection number* of G, denoted by $\Theta_{L,k}(G)$, is the minimum size of the universe of the sets in all k-uniform intersection representations of graph G. As a natural extension, we can assume that the size of sets assign to the vertices are restricted to r different sizes in the set $K = \{k_1, k_2, \ldots, k_r\}$. In this case, we denote the minimum size of the universe of the sets in all such representations with $\Theta_{L,K}(G)$. Now we investigate the uniform case and obtain the similar lower bounds for $\Theta_{L,k}$ and $\Theta_{L,K}$ for various types L.

Proposition 5.1. [6, Proposition 7.9 on Page 142] *If $\mathcal{F}$ is a subfamily of k-subsets of $[l]$, then*

$$I_l(\mathcal{F}, i)I_l(i, t) = \binom{k-t}{i-t} I_l(\mathcal{F}, t).$$

Theorem 5.2. *Let p be a prime number and R be a subset of residues module p with $|R| = r$. If $L = \{l : l \pmod p \in R\}$, then for every graph G,*

$$\mathrm{mr_p}(\overline{G}) \le \binom{\Theta_{L,k}(G)}{r}.$$

Proof. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be the k-uniform family of subsets assigned to the vertices of G by $\Theta_{L,k}$ labels. Let $M_t = A_t(\mathcal{F}, \mathcal{F}) = I_l(\mathcal{F}, t)I_l(\mathcal{F}, t)^T$ be the t-intersection matrix of $\mathcal{F}$, where $0 \le t \le r$. Recall that, M_t is an $n \times n$ matrix, with $\binom{|F_u \cap F_v|}{t}$ in position (u, v).

By Proposition 5.1,

$$I_l(\mathcal{F}, r)I_l(r, t) = \binom{k-t}{r-t} I_l(\mathcal{F}, t).$$

Note that, the column vector space of M_t is a subspace of column vector space of $I_l(\mathcal{F}, t)$. Moreover, if $0 \le t \le r \le k$, then $\binom{k-t}{r-t} \ne 0$. Thus, by the above relation, the column vector space of $I_l(\mathcal{F}, t)$ is a subspace of column vector space of $I_l(\mathcal{F}, r)$.

Now, by Theorem 3.1(i), there are real numbers c_t, $0 \le t \le r$, such that in $\mathbb{Z}_p$ for $M = \sum_{t=0}^{r} c_t A_t(\mathcal{F}, \mathcal{F})$, we have $\mathcal{G}(M) \cong \overline{G}$. By the definition of M, the column vector space of M is a subspace of the vector space spanned by the columns of M_t, $0 \le t \le r$. Hence, it is the subspace of the column vector space of $I_l(\mathcal{F}, r)$. Therefore,

$$\mathrm{rank}(M) \le \mathrm{rank}(I_l(\mathcal{F}, r)) \le \binom{\Theta_{L,k}(G)}{r}.$$

Therefore, by $\mathcal{G}(M) \cong \overline{G}$,

$$\mathrm{mr_p}(\overline{G}) \le \mathrm{rank}(M) \le \binom{\Theta_{L,k}(G)}{r}.$$

□

A natural extension of the uniform representation is a set representation with the restriction on the size of sets to r different sizes. For such a representation, a generalization of Theorem 5.2 is proved in the next theorem.

Theorem 5.3. *If $L = \{l_1, \dots, l_s\}$ and $K = \{k_1, \dots, k_r\}$ are two subsets of non-negative integers, where $k_i > s - r$, $1 \le i \le r$, then for every graph G,*

$$\mathrm{mr}(\overline{G}) \le r \sum_{t=s-r+1}^{s} \binom{\Theta_{L,K}(G)}{t}.$$

Proof. Let $\mathcal{F} = \{F_1, \dots, F_n\}$ be a family of sets corresponding to an optimal (L, K)-intersection representation of G, and $\mathcal{F}_i$, $1 \le i \le r$, be the k_i-uniform subfamily of subsets of $\mathcal{F}$. Suppose that $M_t = A_t(\mathcal{F}, \mathcal{F})$ is the t-intersection matrix of $\mathcal{F}$. In this case, similar to the Theorem 3.1(i), one can prove that there are real numbers c_t, $0 \le t \le s$, such that for $M = \sum_{t=0}^{s} c_t M_t$, $\mathcal{G}(M) \cong \overline{G}$.

For convenience, we denote the column vector spaces of a matrix Q by $C(Q)$, respectively. By the definition of $M_t = I_l(\mathcal{F}, t) I_l(\mathcal{F}, t)^T$,

$$C(M_t) \subseteq C(I_l(\mathcal{F}, t)).$$

Moreover, by Proposition 5.1, we have,

$$I_l(\mathcal{F}_i, s - r + 1) I_l(s - r + 1, t) = \binom{k_i - t}{s - r + 1 - t} I_l(\mathcal{F}_i, t).$$

If $0 \le t \le s - r + 1$, then $t \le s - r + 1 \le k_i$ and $\binom{k_i - t}{s-r+1-t} \ne 0$. Hence, by the above equality, for $0 \le t \le s - r + 1$,

$$C(I_l(\mathcal{F}_i, t)) \subseteq C(I_l(\mathcal{F}_i, s - r + 1)). \tag{3}$$

Now, by the definitions, we have,

$$C(M) \subseteq \sum_{t=0}^{s} C(M_t) \subseteq \sum_{t=0}^{s} C(I_l(\mathcal{F}, t)).$$

Since $\mathcal{F} = \sum_{i=1}^{r} \mathcal{F}_i$,

$$\sum_{t=0}^{s} C(I_l(\mathcal{F}, t)) \subseteq \sum_{t=0}^{s} \sum_{i=1}^{r} C(I_l(\mathcal{F}_i, t)).$$

Also, by relation 3,

$$\sum_{t=0}^{s} \sum_{i=1}^{r} C(I_l(\mathcal{F}_i, t)) \subseteq \sum_{i=1}^{r} \sum_{t=s-r+1}^{s} C(I_l(\mathcal{F}_i, t)).$$

Hence,

$$C(M) \subseteq \sum_{i=1}^{r} \sum_{t=s-r+1}^{s} C(I_l(\mathcal{F}_i, t)).$$

Thus,

$$\begin{aligned} \mathrm{rank}(M) &\leq \textstyle\sum_{j=1}^{r} \sum_{t=s-r+1}^{s} |C(I_l(\mathcal{F}_i, t))| \\ &\leq \textstyle\sum_{i=1}^{r} \sum_{t=s-r+1}^{s} \binom{\Theta_{L,K}(G)}{t} \\ &= r \textstyle\sum_{t=s-r+1}^{s} \binom{\Theta_{L,K}(G)}{t}. \end{aligned}$$

On the other hand, since $\mathcal{G}(M) \cong \overline{G}$,

$$\mathrm{mr}(\overline{G}) \leq \mathrm{rank}(M) \leq r \sum_{t=s-r+1}^{s} \binom{\Theta_{L,K}(G)}{t}.$$

□

By the same argument as in Theorems 5.2 and 5.3, the same lower bounds for the bipartite version can be obtained.

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