On the total restrained domination edge critical graphs

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Abstract

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a total restrained dominating set of $G$ if every vertex in $V$ has a neighbor in $D$ and every vertex in $V - D$ has a neighbor in $V - D$. The cardinality of a minimum total restrained dominating set in $G$ is the total restrained domination number of $G$. In this paper, we define the concept of total restrained domination edge critical graphs, find a lower bound for the total restrained domination number of graphs, and constructively characterize trees having their total restrained domination numbers achieving the lower bound.

Key Words: Domination; Total restrained domination number; Total restrained domination edge critical graphs; Matching; Edge cover; Trees.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n(G)$ and size $|E| = m(G)$. If there is no confusion, then we omit $G$ in these notations and call $G$ an $(n, m)$-graph. The degree of a vertex $v$ in $G$ is the number of vertices adjacent to $v$, and denoted by $deg_G(v)$. A vertex with no neighbor in $G$
is called an isolated vertex. A vertex of degree one in $G$ is called an end vertex, the vertex adjacent to and the edge incident to an end vertex are called a support vertex and a tail, respectively. An edge is called a strong edge if it is not a tail. A path $P$ in $G$ is called an end path of $G$ if $P$ contains an end vertex of $G$ and the degree of each vertex of $P$ in $G$ except end vertices is 2.

A set $D \subseteq V$ is a dominating set of $G$ if every vertex in $V - D$ has a neighbor in $D$. The cardinality of a minimum dominating set of $G$ is the domination number of $G$ and denoted by $\gamma(G)$ (see [5, 6]). If, in addition, the induced subgraph $(D)$ has no isolated vertex, then $D$ is called a total dominating set (TDS). The cardinality of a minimum total dominating set of $G$ is called the total domination number and denoted by $\gamma_t(G)$. The total domination in graphs was introduced by Cockayne et al. in [1] (see also [3, 6, 9]).

Throughout this paper, we assume that $G$ contains no isolated vertices. A set $D \subseteq V$ is a total restrained dominating set of $G$ (TRDS) if $D$ is a TDS of $G$ and also the induced subgraph $(V - D)$ has no isolated vertex. Note that the set $V$ is a TRDS of $G$. The cardinality of a minimum total restrained dominating set of $G$ is called the total restrained domination number of $G$ and denoted by $\gamma_{tr}(G)$. We call a TRDS in graph $G$ of cardinality $\gamma_{tr}(G)$ a $\gamma_{tr}(G) - set$. The concept of total restrained domination was introduced by De-Xiang Ma et al. in [7].

A graph $G$ is said to be total restrained domination edge critical if for every strong edge $e$ in $G$, $\gamma_{tr}(G - e) > \gamma_{tr}(G)$. For simplicity, we call such $G$ a $\gamma_{tr}$-edge critical graph. In this paper, we first characterize $\gamma_{tr}$-edge critical paths, cycles and caterpillars and find necessary and sufficient conditions for a graph to be $\gamma_{tr}$-edge critical. We then proceed to find a lower bound and an upper bound of $\gamma_{tr}(G)$ for $\gamma_{tr}$-edge critical graphs $G$, and hence derive a lower bound of $\gamma_{tr}(G)$ for all $(n, m)$-graphs $G$. Finally we characterize the trees which have their total restrained domination number achieving the lower bound. For unexplained terms and symbols, see [10].

2 Known results

In this section, we state some known results which are useful for proving our main theorems.
Proposition A. [2] Let \( D \) be a TRDS of a graph \( G \) of order \( n \), \( n \geq 3 \). Then every end vertex and every support vertex of \( G \) are in \( D \).

Proposition B. [7] For every integer \( n \), \( n \geq 2 \),

(i) \( \gamma_{tr}(K_n) = \begin{cases} 3 & \text{if } n = 3, \\ 2 & \text{if } n \neq 3; \end{cases} \)

(ii) \( \gamma_{tr}(K_{p,q}) = \begin{cases} p + q & \text{if } \min\{p, q\} = 1, \\ 2 & \text{if } \min\{p, q\} \neq 1; \end{cases} \)

(iii) \( \gamma_{tr}(P_n) = n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor \)

(iv) \( \gamma_{tr}(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor . \)

A tree \( T \) is called a caterpillar if the resulting subgraph of \( T \) obtained by deleting all its end vertices is a path. We call this path the spine of the caterpillar. Let \( T \) be a caterpillar with spine \( v_1...v_s \) and let \( \{u_0 = v_1, u_1, ..., u_{k+1} = v_s\} \) be the ordered set of vertices in \( \{v_1, ..., v_s\} \) with \( \text{deg}_T(u_i) > 2 \), for each \( 1 \leq i \leq k \). We denote the number of internal vertices in \( (u_i, u_{i+1}) \)-path by \( z_i \), \( 0 \leq i \leq k \), and one of the end vertices adjacent to \( u_i \), \( 0 \leq i \leq k+1 \), by \( a_i \).

Proposition C. [2] For every caterpillar \( T \) of order \( n \), \( n \geq 3 \), \( \gamma_{tr}(T) = n - 2 \sum_{i=1}^{k} \left\lfloor \frac{z_i + 2}{4} \right\rfloor . \)

Let \( G \) be a graph. A set \( M \subseteq E \) is called a matching if no two edges in \( M \) are adjacent. The cardinality of a maximum matching in \( G \) is denoted by \( \alpha'(G) \). A set \( L \subseteq E \) is called an edge cover of \( G \) if every vertex of \( G \) is incident to some edge of \( L \). The cardinality of a minimum edge cover is called the edge cover number of \( G \) and denoted by \( \beta'(G) \). Obviously, the edge cover number of a graph is equal to the sum of the edge cover numbers of its components. The well known Gallai identity relating \( \alpha'(G) \) and \( \beta'(G) \) is stated below.

Theorem A. [10] If \( G \) is a graph of order \( n \) without isolated vertices, then \( \alpha'(G) + \beta'(G) = n \).
3 \( \gamma_{tr} \)-edge critical graphs

In this section, we first characterize \( \gamma_{tr} \)-edge critical paths, cycles and caterpillars and provide necessary and sufficient conditions for a graph to be \( \gamma_{tr} \)-edge critical. We then proceed to derive a lower bound and an upper bound for the total restrained domination number of \( \gamma_{tr} \)-edge critical graphs.

It is obvious that every TRDS of a spanning subgraph \( H \) of graph \( G \) is also a TRDS of \( G \). Thus we have:

**Observation 1.** If \( H \) is a spanning subgraph of a graph \( G \), then \( \gamma_{tr}(H) \geq \gamma_{tr}(G) \).

This observation implies that the \( \gamma_{tr}(G) \) is nondecreasing if we delete an edge of \( G \).

**Definition.** A graph \( G \) is a \( \gamma_{tr} \)-edge critical graph if for every strong edge \( e \) of \( G \), \( \gamma_{tr}(G - e) > \gamma_{tr}(G) \).

It is clear that every graph \( G \) contains a \( \gamma_{tr} \)-edge critical spanning subgraph \( H \) with \( \gamma_{tr}(H) = \gamma_{tr}(G) \). This is seen by removing edges in succession, whenever possible, without diminishing the total restrained domination number.

**Remark 1.** The difference \( \gamma_{tr}(G - e) - \gamma_{tr}(G) \) can be arbitrary large. For example, in the graph of Figure 1, \( \gamma_{tr}(G) = k + 3 \) while \( \gamma_{tr}(G - e) = 2k + 4 \), for \( k \geq 1 \). Note that \( D = A_1 \cup A_2 \) is a \( \gamma_{tr}(G) \)-set and \( D' = A_1 \cup A_2 \cup B_1 \cup B_2 \) is a \( \gamma_{tr}(G - e) \)-set, where \( e \) is the dotted edge denoted in graph \( G \).

Suppose that \( G \) is a graph with components \( G_1, G_2, \ldots, G_k \) and for each \( i, 1 \leq i \leq k \), \( D_i \) is a TRDS of \( G_i \). Then the union of \( D_i \) is a TRDS of \( G \). Thus, we have:

**Observation 2.** If \( G \) is a graph with components \( G_1, G_2, \ldots, G_k \), then

\[
\gamma_{tr}(G) = \sum_{i=1}^{k} \gamma_{tr}(G_i).
\]

By Observation 2, the following observation is immediate.
Figure 1: Graph $G$, where $\gamma_{tr}(G) = k + 3$ and $\gamma_{tr}(G - e) = 2k + 4$.

**Observation 3.** A graph $G$ is $\gamma_{tr}$-edge critical if and only if each component of $G$ is $\gamma_{tr}$-edge critical.

**Theorem 1.**

(i) The path $P_n, n \geq 2$, is $\gamma_{tr}$-edge critical if and only if $n \equiv 2$ or $3 \pmod{4}$.

(ii) The cycle $C_n, n \geq 3$, is $\gamma_{tr}$-edge critical if and only if $n \equiv 0$ or $1 \pmod{4}$.

(iii) The caterpillar $T$ is $\gamma_{tr}$-edge critical if and only if for each $i, 0 \leq i \leq k$, $z_i \equiv 2$ or $3 \pmod{4}$ (see page 2 for the definition of $z_i$).

**Proof.** (i) Consider the path $P_n$ of order $n$ and assume that $n \equiv 0$ or $1 \pmod{4}$. Let $e$ be an edge adjacent to a tail. Then $P_n - e$ is a graph with two components $P_2$ and $P_{n-2}$. By Proposition B(iii) and Observation 2,

$$\gamma_{tr}(P_n - e) = \gamma_{tr}(P_2) + \gamma_{tr}(P_{n-2})$$

$$= 2 + (n - 2) - 2 \left\lfloor \frac{(n - 2) - 2}{4} \right\rfloor$$

$$= n - 2 \left\lfloor \frac{n - 4}{4} \right\rfloor.$$

As $n \equiv 0$ or $1 \pmod{4}$, we have $\left\lfloor \frac{n-4}{4} \right\rfloor = \left\lfloor \frac{n-2}{4} \right\rfloor$, and so

$$\gamma_{tr}(P_n - e) = n - 2 \left\lfloor \frac{n - 2}{4} \right\rfloor = \gamma_{tr}(P_n).$$
Thus, if $n \equiv 0$ or 1 (mod 4), then $P_n$ is not $\gamma_{tr}$-edge critical.

Now suppose that $n \equiv 2$ or 3 (mod 4). Let $e$ be a strong edge of $P_n$. Then $P_n - e$ is a graph with two components $P_{n_1}$ and $P_{n_2}$, such that $n_1 + n_2 = n$. By Proposition B(iii) and Observation 2,

$$
\gamma_{tr}(P_n - e) = \gamma_{tr}(P_{n_1}) + \gamma_{tr}(P_{n_2})
$$

$$
= n - 2 \left( \left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor \right)
$$

Assume that at least one of $n_1$ or $n_2$ is congruent to 0 or 1 modulo 4 (say, $n_1 \equiv 0$ or 1 (mod 4), and so $\left\lfloor \frac{n_1 - 2}{4} \right\rfloor = \left\lfloor \frac{n_1 - 4}{4} \right\rfloor$). Then

$$
\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor = \left\lfloor \frac{n_1 - 4}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor
$$

$$
\leq \frac{n_1 - 4}{4} + \frac{n_2 - 2}{4}
$$

$$
= \frac{n - 2 - 4}{4} = \frac{n - 2}{4} - 1
$$

$$
< \left\lfloor \frac{n - 2}{4} \right\rfloor,
$$

and so $n - 2 \left( \left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor \right) > n - 2 \left\lfloor \frac{n - 2}{4} \right\rfloor$; i.e., $\gamma_{tr}(P_n - e) > \gamma_{tr}(P_n)$.

Assume now that $n_1$ and $n_2$ are congruent to 3 modulo 4. In this case, $\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor = \left\lfloor \frac{n - 2}{4} \right\rfloor - 1$, and it can be easily observed that $\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor < \left\lfloor \frac{n - 2}{4} \right\rfloor$; i.e., $\gamma_{tr}(P_n - e) > \gamma_{tr}(P_n)$.

(ii) As $C_n - e$ is $P_n$ for any edge $e$ in $C_n$, by Proposition B(iv), $C_n$ is $\gamma_{tr}$-edge critical if and only if

$$
n - 2 \left\lfloor \frac{n - 2}{4} \right\rfloor = \gamma_{tr}(P_n) > \gamma_{tr}(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor.
$$

The inequality above holds if and only if $\left\lfloor \frac{n - 2}{4} \right\rfloor < \left\lfloor \frac{n}{4} \right\rfloor$, i.e., $n \equiv 0$ or 1 (mod 4).

(iii) By Proposition A, it can be seen that the caterpillar $T$ is a $\gamma_{tr}$-edge critical graph if and only if the $(a_i, a_{i+1})$-paths are $\gamma_{tr}$-edge critical,
for each $i$, $0 \leq i \leq k$. By the first part above, the latter holds if and only if $z_i + 4 \equiv 2$ or $3 \pmod{4}$. Thus, $T$ is $\gamma_{cr}$-edge critical if and only if for each $i$, $0 \leq i \leq k$, $z_i \equiv 2$ or $3 \pmod{4}$.

\textbf{Theorem 2.} Let $G$ be a graph. Then $G$ is $\gamma_{cr}$-edge critical if and only if every $\gamma_{cr}(G)$-set $D$ satisfies each of the following conditions:

1. Every component of $\langle D \rangle$ and $\langle V - D \rangle$ is a star.
2. Every vertex in $V - D$ has exactly one neighbor in $D$.

Note. Condition (2) implies that the number of edges between $D$ and $V - D$ is equal to $n - \gamma_{cr}(G)$.

\textbf{Proof.} Suppose that $G$ is a $\gamma_{cr}$-edge critical graph and $D$ is a $\gamma_{cr}(G)$-set.

1. If $\langle D \rangle$ or $\langle V - D \rangle$ has a strong edge, then $D$ is a TRDS for the graph obtained from $G$ by deleting the strong edge. This contradicts the fact that $G$ is $\gamma_{cr}$-edge critical. Thus, every component of $\langle D \rangle$ and $\langle V - D \rangle$ is a star.
2. Every vertex in $V - D$ has more than one neighbor in $D$. If a vertex $v$ in $V - D$ has more than one neighbor in $D$, say $u_1$ and $u_2$, then $D$ is a TRDS of the graph $G - u_1v$, a contradiction. Thus, condition (2) holds.

We now prove the sufficiency by contradiction. Assume that every $\gamma_{cr}(G)$-set satisfies the two conditions, but $G$ is not $\gamma_{cr}$-edge critical. Let $H$ be a $\gamma_{cr}$-edge critical proper spanning subgraph of $G$ such that $\gamma_{cr}(H) = \gamma_{cr}(G)$. Suppose that $D$ is a $\gamma_{cr}(H)$-set. By the above necessity conditions, $D$ satisfies conditions (1) and (2) in $H$. Observe that $D$ is also a $\gamma_{cr}(G)$-set, but now $D$ no longer satisfies the conditions in $G$, as $G$ contains at least one edge not in $H$. This contradiction shows that $G$ is $\gamma_{cr}$-edge critical.

\textbf{Corollary 1.} Let $G$ be an $(n, m)$-graph. If $G$ is $\gamma_{cr}$-edge critical, then

$$\frac{3n}{2} - m \leq \gamma_{cr}(G) \leq 2n - m - 2.$$

\textbf{Proof.} Let $D$ be a $\gamma_{cr}(G)$-set. By Theorem 2, the number of edges with one end in $D$ and another one in $V - D$ is equal to $n - \gamma_{cr}(G)$. As $\langle D \rangle$ and $\langle V - D \rangle$ are forests, the number of edges in $\langle D \rangle$ and $\langle V - D \rangle$ does not exceed $|D| - 1$ and $|V - D| - 1$, respectively. Thus,

$$m \leq (|D| - 1) + (|V - D| - 1) + (n - \gamma_{cr}(G))$$

$$= (\gamma_{cr}(G) - 1) + (n - \gamma_{cr}(G)) + (n - \gamma_{cr}(G))$$

$$= 2n - \gamma_{cr}(G) - 2,$$
and so
\[ \gamma_{tr}(G) \leq 2n - m - 2. \]

On the other hand, as the degree of every vertex in \( \langle D \rangle \) and \( \langle V - D \rangle \) is at least one, we have
\[
m \geq \frac{|D|}{2} + \frac{|V - D|}{2} + n - \gamma_{tr}(G) \\
= \frac{\gamma_{tr}(G)}{2} + \frac{n - \gamma_{tr}(G)}{2} + n - \gamma_{tr}(G) \\
= \frac{3n}{2} - \gamma_{tr}(G),
\]
i.e.,
\[
\frac{3n}{2} - m \leq \gamma_{tr}(G).
\]

4 Total restrained domination number of graphs

In this section, we find some bounds for the total restrained domination number of graphs.

**Lemma 1.** Let \( D \) be a \( \gamma_{tr}(G) \)-set of a \( \gamma_{tr} \)-edge critical graph \( G \). If \( k \) and \( k' \) are the numbers of components in \( \langle D \rangle \) and \( \langle V - D \rangle \), respectively, then
\[ \gamma(G) \leq k + k' \leq \alpha'(G). \]

**Proof.** By Theorem 2, every component of \( \langle D \rangle \) and \( \langle V - D \rangle \) is a star. Let \( A \) be the set of the centers of these stars. Then \( A \) is a dominating set of \( G \) and \( |A| = k + k' \). Hence
\[ \gamma(G) \leq |A| = k + k'. \]

Form a set \( B \subseteq E \) by selecting an edge from each component of \( \langle D \rangle \) and \( \langle V - D \rangle \). Then \( B \) is a matching of \( G \), and so by above inequality
\[ \gamma(G) \leq k + k' = |B| \leq \alpha'(G). \]
Remark 2. Suppose that $G$ is a graph and $D$ is a subset of $V$ such that each component of $(D)$ and $(V - D)$ is a star. Denote the set of edges between $D$ and $V - D$ by $F_D(G)$ and let $f_D(G) = |F_D(G)|$. Now we construct a bipartite multigraph $G^*_D$ with partite sets $X$ and $Y$ from $G$ with respect to $D$ as follows. Every vertex in $X$ corresponds to a component of $(D)$ and every vertex in $Y$ corresponds to a component of $(V - D)$. Let $k$ and $k'$ be the numbers of components in $(D)$ and $(V - D)$, respectively; so $|X| = k$ and $|Y| = k'$. Corresponding to every edge in $G$ joining a component of $(D)$ and a component of $(V - D)$, there is an edge in $G^*_D$ joining the two vertices corresponding to the components (note that $G^*_D$ may contain multiple edges). Then $G^*_D$ is an $(n^*, m^*)$-multigraph, where $n^* = n(G^*_D) = k + k'$ and $m^* = m(G^*_D) = f_D(G)$.

Referring to the notations in Remark 2, we have:

Lemma 2. 

$$m(G^*_D) = n(G^*_D) - (n(G) - m(G)).$$

Proof. We prove the equality by induction on $f_D(G)$. Assume $f_D(G) = 0$. Then $G$ is a forest with $k + k'$ components, and so $m(G) = n(G) - (k + k')$. Hence $n(G) - m(G) = k + k' = n(G^*_D) - m(G^*_D)$.

Assume that $f_D(G) > 0$ and the equality holds for every graph $H$ with $f_D(H) < f_D(G)$. Suppose that $H$ is a graph obtained from $G$ by deleting an edge of $F_D(G)$. Then $f_D(H) = f_D(G) - 1 < f_D(G)$, and by the induction hypothesis, $m(H^*_D) = n(H^*_D) - (n(H) - m(H))$. Since $m(H^*_D) = m(G^*_D) - 1$, $n(H^*_D) = n(G^*_D)$, $m(H) = m(G) - 1$ and $n(H) = n(G)$, we have $m(G^*_D) = n(G^*_D) - (n(G) - m(G))$, as desired.

Theorem 3. For every $\alpha_{tr}-$critical $(n, m)$-graph $G$,

$$\beta'(G) + n - m \leq \gamma_{tr}(G) \leq 2n - m - \gamma(G).$$

Proof. Let $D$ be a $\gamma_{tr}(G)$-set and $G^*_D$ be the corresponding $(n^*, m^*)$-multigraph constructed from $G$ as described in Remark 2. By Theorem 2, $m^* = f_D(G) = n - \gamma_{tr}(G)$, and by Lemma 2, $n^* - (n - m) = m^*$. Hence

$$k + k' - (n - m) = n^* - (n - m) = m^* = n - \gamma_{tr}(G),$$

and so

$$\gamma_{tr}(G) = n - (k + k') + (n - m).$$

This equality and the inequalities in Lemma 1 imply that

$$n - \alpha'(G) + (n - m) \leq \gamma_{tr}(G) \leq n - \gamma(G) + (n - m).$$
Now, by Theorem A, we have
\[
\beta'(G) + n - m \leq \gamma_{tr}(G) \leq 2n - m - \gamma(G).
\]

**Remark 3.** The above bounds are sharp, as stars are $\gamma_{tr}$-edge critical graphs and their $\gamma_{tr}$ achieve both lower and upper bounds above.

**Corollary 2.** If $G$ is an $(n, m)$-graph, then $\gamma_{tr}(G) \geq \beta'(G) + n - m$.

**Proof.** Suppose that $H$ is a $\gamma_{tr}$-edge critical spanning subgraph of $G$ such that $\gamma_{tr}(H) = \gamma_{tr}(G)$. Since $H$ is a spanning subgraph of $G$, each edge cover of $H$ is an edge cover of $G$, so $\beta'(G) \leq \beta'(H)$. Hence by Theorem 3,
\[
\beta'(G) + n - m \leq \beta'(H) + n(H) - m(H) \leq \gamma_{tr}(H) = \gamma_{tr}(G).
\]

**Remark 4.** In [2] it is proved that if $G$ is an $(n, m)$-graph, then
\[
\gamma_{tr}(G) \geq \frac{3n}{2} - m;
\]
and in [4] it is proved that if $T$ is a tree of order $n$, then
\[
\gamma_{tr}(T) \geq \left\lceil \frac{n + 2}{2} \right\rceil.
\]
Since for every graph $G$ of order $n$, $\frac{n}{2} \leq \left\lceil \frac{n + 1}{2} \right\rceil \leq \beta'(G)$, the lower bound obtained in Corollary 2 is sharper than the above two.

**Theorem 4.** If $G$ is an $(n, m)$-graph such that $\gamma_{tr}(G) = \beta'(G) + n - m$, then $G$ is $\gamma_{tr}$-edge critical.

**Proof.** We prove the statement by contradiction. Suppose that $G$ is not $\gamma_{tr}$-edge critical. Then there is an edge, say $e$, such that $\gamma_{tr}(G - e) = \gamma_{tr}(G)$. By Corollary 2 and the hypothesis,
\[
\begin{align*}
\gamma_{tr}(G) &= \beta'(G) + n - m \leq \beta'(G - e) + n - m \\
&= \beta'(G - e) + n(G - e) - (m(G - e) + 1) \\
&\leq \gamma_{tr}(G - e) - 1 = \gamma_{tr}(G) - 1,
\end{align*}
\]
a contradiction.
Remark 5. For every integer $k > 0$ there exists a graph $G$ such that $\gamma_{tr}(G) - \beta'(G) = k + 1$. For instance, in the graph $G$ of Figure 2, the set $D = \bigcup_{i=1}^{2k+2} A_i$ is a $\gamma_{tr}(G)$-set with $|D| = 5k + 2$ and the set of bold edges is an edge cover of size $4k + 1$. Moreover, note that graph $G$ is $\gamma_{tr}$-edge critical. So this example shows that the converse of Theorem 4 is not true.

Figure 2: Graph $G$, where $\gamma_{tr}(G) - \beta'(G) = k + 1$.

5 Characterization of trees with minimum $\gamma_{tr}$

It follows from Corollary 2 that if $T$ is a tree, then $\gamma_{tr}(T) \geq \beta'(T) + 1$. In this final section, we characterize all trees $T$ such that $\gamma_{tr}(T) = \beta'(T) + 1$. We first present some useful lemmas.

Lemma 3. Suppose that $T$ and $T'$ are two trees such that for some integer $k$, $\gamma_{tr}(T') \leq \gamma_{tr}(T) + k$ and $\beta'(T) \leq \beta'(T') - k$. If $\gamma_{tr}(T) = \beta'(T) + 1$, then $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) + k$.

Proof. By Corollary 2 and the hypothesis, we have

$$\beta'(T') + 1 \leq \gamma_{tr}(T') \leq \gamma_{tr}(T) + k$$

$$= (\beta'(T) + 1) + k = (\beta'(T) + k) + 1 \leq \beta'(T') + 1.$$ 

Hence $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) + k$.

Lemma 4. Suppose that $T$ and $T'$ are two trees such that for some integer $k$, $\gamma_{tr}(T') \leq \gamma_{tr}(T) - k$ and $\beta'(T) \leq \beta'(T') + k$. If $\gamma_{tr}(T) = \beta'(T) + 1$, then $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) - k$. 

11
Proof. By Corollary 2 and the hypothesis, we have
\[
\beta'(T') + 1 \leq \gamma_{tr}(T') \leq \gamma_{tr}(T) - k
\]
\[
= ((\beta(T) + 1) - k = (\beta(T) - k) + 1 \leq \beta'(T') + 1.
\]
Hence \(\gamma_{tr}(T') = \beta'(T') + 1\) and \(\gamma_{tr}(T') = \gamma_{tr}(T) - k\).

Lemma 5. Let \(T\) be a tree with \(\gamma_{tr}(T) = \beta'(T) + 1\) and \(P\) be an end path with \(k\) vertices in \(T\). If \(D\) is a \(\gamma_{tr}(T)\)-set such that \(D' = D - V(P)\) is a TRDS for \(T' = T - V(P)\), then at most \(\lfloor \frac{k+1}{2} \rfloor\) vertices of \(P\) belong to \(D\).

Proof. Suppose that this is not true; i.e., \(D\) contains at least \(\lfloor \frac{k+1}{2} \rfloor + 1\) vertices of \(P\). By Corollary 2,
\[
\beta'(T') + 1 \leq \gamma_{tr}(T').
\]
Since \(D' = D - V(P)\) is a TRDS of \(T'\),
\[
\gamma_{tr}(T') \leq |D'| \leq |D| - \left( \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \right) = \gamma_{tr}(T) - \left\lfloor \frac{k+1}{2} \right\rfloor - 1.
\]
The union of an edge cover of \(P\) and an edge cover of \(T'\) is an edge cover of \(T\) and \(\beta'(P) = \left\lfloor \frac{k+1}{2} \right\rfloor\). Thus
\[
\beta'(T) \leq \beta'(T') + \left\lfloor \frac{k+1}{2} \right\rfloor.
\]
Now we have
\[
\beta'(T') + 1 \leq \gamma_{tr}(T') \leq \gamma_{tr}(T) - \left\lfloor \frac{k+1}{2} \right\rfloor - 1
\]
\[
= \beta'(T) + 1 - \left\lfloor \frac{k+1}{2} \right\rfloor - 1 = \beta'(T) - \left\lfloor \frac{k+1}{2} \right\rfloor
\]
\[
\leq (\beta'(T') + \left\lfloor \frac{k+1}{2} \right\rfloor) - \left\lfloor \frac{k+1}{2} \right\rfloor = \beta'(T'),
\]
a contradiction. This shows that \(D\) contains at most \(\left\lfloor \frac{k+1}{2} \right\rfloor\) vertices of \(P\).

Now we construct a family \(\Phi\) of trees recursively as follows:

(i) Let \(P_2\) be in \(\Phi\).
(ii) Let \(T \in \Phi\) and \(D\) be a \(\gamma_{tr}(T)\)-set. Then \(T' \in \Phi\) if \(T'\) is a tree constructed from \(T\) by performing one of the following operations.
O1. Add a new vertex \( t \) to \( T \) and join \( t \) to a support vertex in \( T \). Let \( D' := D \cup \{ t \} \).

O2. Add a new path \( abcd \) to \( T \) and join vertex \( a \) to a vertex \( s \) in \( D \). Let \( D' := D \cup \{ c, d \} \).

O3. Let \( abcd \) be an end path in \( T \) such that \( a \notin D \) and \( b, c, d \in D \). Add a new path \( tx \) to \( T \), and join \( t \) to vertex \( a \). Let \( D' := (D - \{ b \}) \cup \{ t, x \} \).

O4. Let \( abcd \) be an end path in \( T \) such that \( a \notin D \). Add a new path \( txy \) to \( T \) and join \( t \) to \( a \). Let \( D' := D \cup \{ x, y \} \).

In the following lemma, we show that \( D' \) is a \( \gamma_{tr}(T') \)-set and hence \( \Phi \) can be constructed recursively.

**Lemma 6.** Let \( T \) be a tree such that \( \gamma_{tr}(T) = \beta'(T) + 1 \) and \( T' \) constructed from \( T \) by one of the operations above. Then \( \gamma_{tr}(T') = \beta'(T') + 1 \) and \( D' \) is a \( \gamma_{tr}(T') \)-set.

**Proof.** We first show that if we perform each of the operations above, then \( T \) and \( T' \) satisfy the hypothesis of Lemma 3 for some \( k \). Hence we can conclude that \( \gamma_{tr}(T') = \beta'(T') + 1 \). To see this, let \( M' \) be an edge cover of \( T' \).

Operation \( O_1 \). By Proposition A, we have every support vertex is in \( D \), so it is obvious that \( D' \) is a TRDS of \( T' \). Thus \( \gamma_{tr}(T') \leq |D'| = \gamma_{tr}(T) + 1 \). Suppose that \( M \) is obtained from \( M' \) by deleting the edge incident to \( t \) (note that each edge incident to an end vertex belongs to \( M' \)). The set \( M \) is an edge cover for \( T \); so \( \beta'(T) \leq \beta'(T') - 1 \). In this case, \( k = 1 \) and we are done.

Operation \( O_2 \). Similarly, for this operation, we have \( \gamma_{tr}(T') \leq |D'| = \gamma_{tr}(T) + 2 \). Suppose that \( M \) is obtained from \( M' \) by deleting the edges incident to the vertices \( b \) and \( d \). Since \( b \) and \( d \) are not adjacent, there are at least two such edges. Moreover if edge \( as \) belongs to \( M \), then we substitute \( as \) with an edge of \( T \) incident to \( s \) to get an edge cover for \( T \). So \( \beta'(T) \leq \beta'(T') - 2 \). Hence, in this case, \( k = 2 \) and we are done.

Operation \( O_3 \). For this operation, we have \( k = 1 \), and the argument is similar to the above.

Operation \( O_4 \). Similarly, \( \gamma_{tr}(T') \leq \gamma_{tr}(T) + 2 \). If \( at, ab \in M' \), then we can substitute \( at \) with \( tx \) and get a new edge cover of \( T' \). Hence by symmetry of edges \( ab \) and \( at \), without loss of generality we may assume \( at \notin M' \). Thus \( tx \in M' \), also we know that \( xy \in M' \), and so \( M' - \{ tx, xy \} \) is an edge.
cover for $T$ of size $\beta'(T') - 2$. Hence $\beta'(T) \leq \beta'(T') - 2$ and we have $k = 2$, and the desired result can be obtained.

For the second part of the lemma, it is seen that in each case $D'$ is a TRDS of $T'$. Moreover, in each case for chosen $k$, we have $|D'| = \gamma_{tr}(T) + k$. On the other hand, by Lemma 3, $\gamma_{tr}(T') = \gamma_{tr}(T) + k$. Thus $|D'| = \gamma_{tr}(T')$ and so $D'$ is a $\gamma_{tr}(T')$-set.

\textbf{Theorem 5.} The set $\Phi$ is the set of all trees $T$ with $\gamma_{tr}(T) = \beta'(T) + 1$.

\textbf{Proof.} Obviously $\gamma_{tr}(P_2) = 2 = \beta'(P_2) + 1$. Thus by Lemma 6 and using the induction on the number of the operations, for every tree $T$ in $\Phi$, we have $\gamma_{tr}(T) = \beta'(T) + 1$.

We now show that every tree $T$ of order $n$ with $\gamma_{tr}(T) = \beta'(T) + 1$ is contained in $\Phi$. Our proof is by induction on $n$. For $n = 2$, we have $T = P_2$, and $P_2 \in \Phi$. Suppose that $n \geq 3$ and the statement is true for all trees of order less than $n$. Our strategy is to find some proper subtree of $T$, say $T'$, that satisfies the hypothesis of Lemma 4. Hence $\gamma_{tr}(T') = \beta'(T') + 1$ and by the induction hypothesis, $T'$ belongs to $\Phi$. Moreover, we find $T'$ such that $T$ can be constructed from $T'$ by performing one of the operations $O_1, \ldots, O_4$, and conclude that $T \in \Phi$.

Thus, let $T$ be a tree of order $n \geq 3$ with $\gamma_{tr}(T) = \beta'(T) + 1$. Note that, by Theorem 4, $T$ is $\gamma_{tr}$-edge critical. Suppose that $D$ is a $\gamma_{tr}(T)$-set, $P$ is the longest path in $T$ and $c$ is a support vertex in $P$.

If $\deg_G(c) > 2$, then $c$ is adjacent to two end vertices, say $t$ and $d$. By Proposition A, the vertices $t, d$ and $c$ are in $D$. Since $D' = D - \{t\}$ is a TRDS in $T' = T - \{t\}$, $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 1$. On the other hand, the union of an edge cover of $T' = T - \{t\}$ and edge $ct$ is an edge cover of $T$, so $\beta'(T) \leq \beta'(T') + 1$. On the other hand, the union of a $\gamma_{tr}(T')$-set and $\{t\}$ is a TRDS of $T$. Thus $\gamma_{tr}(T) \leq \gamma_{tr}(T') + 1$, and so $\gamma_{tr}(T) = \gamma_{tr}(T') + 1$. Therefore the tree $T'$ is a desired subtree of $T$ from which $T$ can be constructed by $O_1$.

Assume now that $\deg_T(c) = 2$. Then $c$ is adjacent to an end vertex, say $d$ and vertex, say $b$. If $\deg_T(b) = 1$, then $T = P_2$ and $P_2 \in \Phi$. Assume that $\deg_T(b) \geq 2$. Then we have the following two cases to consider.

Case 1. $\deg_T(b) > 2$.

In this case, $b$ has a neighbor not in $P$, say $t$. By our choice of $P$, it is obvious that the length (say $l$) of the longest path $bt \ldots$ beginning with $bt$ is at most two. By Proposition A, the vertices $c, d, t$ and the neighbors of $t$ other than $b$ (if there exist) are in $D$. Thus, for $l = 1$ and $l = 2$, $\gamma_{tr}(T - bc) = \gamma_{tr}(T)$, which contradicts that $T$ is $\gamma_{tr}$-edge critical.

Case 2. $\deg_T(b) = 2$.

Case 2.
In this case, let the neighbors of $b$ be vertices $a$ and $c$. If $\deg_T(a) = 1$, then $T = P_a$, while $\gamma_{tr}(P_a) \neq \beta'(P_a) + 1$. Thus, we consider the following two subcases.

Case 2.1. $\deg_T(a) > 2$.
Assume that $t$ is a neighbor of $a$ not in $P$. Let $l$ be the length of longest path $at \ldots$ beginning with $at$. Then, by the choice of $P$, it is obvious that $l \leq 3$. The following three cases can happen.

Case 2.1.1. $l = 1$.
By Proposition A, the vertices $a, c$ and $d$ are in $D$, so $b$ is also in $D$. This is a contradiction for, by Theorem 2, every component of $\langle D \rangle$ is a star.

Case 2.1.2. $l = 2$.
If $x$ is an end vertex adjacent to $t$, then by Proposition A, vertices $c, d, t$ and $x$ are in $D$. If $b \in D$, then $a$ has two neighbors in $D$, which, by Theorem 2, contradicts that $T$ is a $\gamma_{tr}$-edge critical graph. Hence $b \in V - D$, and since it should not be an isolated vertex in $\langle V - D \rangle$, we have $a \not\in D$. In this case, let $T' = T - \{t, x\}$. It can be seen that $T$ can be constructed from $T'$ by performing $O_3$. Moreover, it can be easily checked that the union of an edge cover of $T'$ and the edge $tx$ is an edge cover of $T$; so $\beta'(T) \leq \beta'(T') + 1$. On the other hand, $(D \cup \{b\}) - \{x, t\}$ is a TRDS of $T'$ (note that $\deg_T(a) > 2$ and $T$ is $\gamma_{tr}$-edge critical, hence by Theorem 2 all neighbors of $a$ except $t$ are in $V - D$); so $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 1$, and we are done in this case.

Case 2.1.3. $l = 3$
Let $atxy$ be a longest path beginning with $at$ of length 3. Note that the path obtained by substituting the subpath $atxy$ with subpath $abcd$ in $P$ is also a longest path in $T$. So by symmetry, we may assume that $\deg_T(t) = 2$ and $\deg_T(x) = 2$. By Proposition A, the vertices $c, d, x$ and $y$ are in $D$. If $b$ and $t$ both belong to $D$, then $a$ has two neighbors in $D$, which, by Theorem 2, contradicts that $T$ is $\gamma_{tr}$-edge critical. Hence at least one of $b$ and $t$ is in $V - D$, say $t \in V - D$. Since there is no isolated vertex in $(V - D)$ and $x \in D$, we have $a \not\in D$. In this case, let $T' = T - \{t, x, y\}$. Then $T$ can be constructed from $T'$ by performing $O_4$. Moreover, it can be easily seen that the union of an edge cover of $T'$ and the set $\{tx, xy\}$ is an edge cover of $T$; so $\beta'(T) \leq \beta'(T') + 2$. On the other hand, $D - \{x, y\}$ is a TRDS of $T'$ and so $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 2$, and we are done in this case.

Case 2.2. $\deg_T(a) = 2$.
In this case, we denote the neighbors of $a$ by $b$ and $s$. By Proposition A, vertices $c$ and $d$ should be in $D$.
If $b \not\in D$, then since $(V - D)$ contains no isolated vertex, $a \not\in D$ and $s \in D$ to dominate $a$. In this case, let $T' = T - \{a, b, c, d\}$. Then $T$ can be
constructed from $T'$ by performing $O_2$. It can be easily shown that $T$ and $T'$ satisfy the conditions of Lemma 4 for $k = 2$.

If $b \in D$, then $a \in V - D$, because, by Theorem 2, every component of $(D)$ is a star. Since there is no isolated vertex in $(V - D)$, $a \in V - D$ implies that $s \not\in D$. If $deg_T(s) > 2$, let $T' = T - \{a, b, c, d\}$, then the set $D - \{b, c, d\}$ is a $\gamma_{tr}(T')$-set (note that, by Theorem 2, all neighbors of $s$ except one are in $V - D$), while $D$ contains three vertices of the end path $abcd$ in $T$. This contradicts Lemma 5. Thus $deg_T(s) \neq 2$. However, in the case that $deg_T(s) = 1$, we have $T = P_5$, while $\gamma_{tr}(P_5) \neq \beta'(P_5) + 1$. Hence $deg_T(s) = 2$. Furthermore since $a \not\in D$, the only other neighbor of $s$ is in $D$. So $(D - \{b\}) \cup \{s\}$ is also a $\gamma_{tr}(T)$-set which does not contain $b$. We are done so long as $b \not\in D$. 

References


