Graphs and Combinatorics
DOI 10.1007/s00373-012-1189-4

ORIGINAL PAPER

An Upper Bound for the Total Restrained Domination Number of Graphs

Khee M. Koh · Zeinab Maleki · Behnaz Omoomi

Received: 19 June 2009 / Revised: 8 May 2012

Abstract Let G be a graph with vertex set V. A set $D \subseteq V$ is a total restrained dominating set of G if every vertex in V has a neighbor in D and every vertex in $V \setminus D$ has a neighbor in $V \setminus D$. The minimum cardinality of a total restrained dominating set of G is called the total restrained domination number of G, and is denoted by $\gamma_{tr}(G)$. In this paper, we prove that if G is a connected graph of order $n \geq 4$ and minimum degree at least two, then $\gamma_{tr}(G) \leq n - \sqrt[3]{\frac{n}{4}}$.

Keywords Total restrained domination number · Total restrained dominating set · Independent set · Matching · Probabilistic method · Open packing

1 Introduction

Let $G = (V, E)$ be a simple graph of order $n(G)$ and size $m(G)$. The *degree* of a vertex v in G is the number of vertices adjacent to v, and denoted by $deg_G(v)$. A vertex with no neighbor in G is called an *isolated vertex*. A vertex of degree one in G is called an *end vertex*, and the vertex adjacent to an end vertex is called a *support vertex*. The minimum degree and the maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If there is no confusion, we omit G in these

K. M. Koh (✉)
Department of Mathematics, National University of Singapore, Singapore 119076, Singapore
e-mail: matkohkm@nus.edu.sg

Z. Maleki · B. Omoomi
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: zmaleki@math.iut.ac.ir

B. Omoomi
e-mail: bomoomi@cc.iut.ac.ir

Published online: 30 May 2012

notations. A graph $G' = (V', E')$ is called a *subgraph* of G and denoted by $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G and denoted by $\langle V' \rangle$. For a subgraph G' of G, $G \backslash G'$ is obtained from G by deleting all the vertices of G' and their incident edges. The *open neighborhood* of v is the set $N_G(v) := \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. For a set $X \subseteq V$, $N_G(X) = \cup_{v \in X} N_G(v)$ and $N_G[X] = \cup_{v \in X} N_G[v]$.

Let $X, Y \subseteq V$. We say X dominates the set Y if $Y \subseteq N_G(X)$. A set $D \subseteq V$ is a *dominating set* (DS) of G if D dominates $V \backslash D$, i.e., every vertex in $V \backslash D$ has a neighbor in D. The minimum cardinality of a dominating set of G is the *domination number* of G and denoted by $\gamma(G)$ (see [4,5]). If, in addition, the induced subgraph $\langle D \rangle$ has no isolated vertex, then D is called a *total dominating set* (TDS) of G. The minimum cardinality of a TDS of G is called the *total domination number* and denoted by $\gamma_t(G)$. The notion of total domination in graphs was introduced by Cockayne et al. [1] (see also [3,4,6,11]). Further, if D is a dominating set and the induced subgraph $\langle V \setminus D \rangle$ has no isolated vertex, then D is called a *restrained dominating set* (RDS) of G. The minimum cardinality of a RDS of G is called the *restrained domination number* and denoted by $\gamma_r(G)$. The notion of restrained domination in graphs was introduced by Telle and Proskurowski implicitly in [12].

Throughout this paper, we assume that G is a connected graph. A set $D \subseteq V$ is a *total restrained dominating set* (TRDS) of G if D is both a TDS and a RDS of G. Note that the set V is a TRDS of G. The minimum cardinality of a TRDS of G is called the *total restrained domination number* of G and denoted by $\gamma_{tr}(G)$. We call a TRDS of cardinality $\gamma_{tr}(G)$ a $\gamma_{tr}(G) - set$. The concept of the total restrained domination was also introduced by Telle and Proskurowski implicitly in [12] and was formally presented in graph theory by Ma et al. [10] (see also [2,7–9]).

We now state some known results which are relevant to our work in this paper. For unexplained terms and symbols, see [13].

Proposition 1 ([2]) *Every end vertex and support vertex in a graph G are in every TRDS of G.*

Proposition 2 ([10]) *For path P_n and cycle C_n of order n,*

(i) $\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor,\ n \geq 2;$

(ii) $\gamma_{tr}(C_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor,\ n \geq 3.$

In [10], it is proved that the decision problem of existence a TRDS of size k is NP-complete. Hence, it is of interest to provide bounds for this number. Two known upper bounds are shown below.

Theorem 1 ([2]) *If G is a connected graph of order n and minimum degree δ such that $2 \leq \delta \leq n - 2$, then*

$$\gamma_{tr}(G) \leq n - \delta.$$

Theorem 2 ([7]) *If G is a connected graph of order n, maximum degree Δ and minimum degree δ, where* $2 \leq \delta \leq \Delta \leq n-2$*, then*

$$\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1.$$

The bounds in the above two theorems are expressed in terms of $n(G)$ and, $\delta(G)$ or $\Delta(G)$. In this paper, we shall apply these two theorems to establish the following result, which provides an upper bound for $\gamma_{tr}(G)$ solely in terms of $n(G)$.

Theorem 3 *If G is a connected graph of order n,* $n \geq 4$*, and minimum degree* $\delta \geq 2$*, then*

$$\gamma_{tr}(G) \leq n - \sqrt[3]{\frac{n}{4}}.$$

2 Preliminaries

We first present in this section a lemma, and some concepts and notations, which will be used to prove the main result in the next section.

Lemma 1 *Let G be a connected graph with* $\delta \geq 2$*, and path P be a component of order* $l \geq 3$ *in* $\langle S \rangle$*, where* $S \subseteq V(G)$*. Let* $G' := G \backslash P$*. Then* $\gamma_{tr}(G) \leq \gamma_{tr}(G') + \frac{l}{2} + 1$*.*

Proof Let $P := x_1 \ldots x_l$ and D' be a TRDS of $G \backslash P$. Suppose that the vertices x and y are neighbors of x_1 and x_l in $G \backslash P$, respectively. We show that we can add $\frac{l}{2} + 1$ vertices of P to D' to obtain a TRDS of G. One of the following three cases may occur.

Case 1 $x, y \in D'$. In this case, we add two paths uvx_1 and $x_l st$ to path $x_1 x_2 \ldots x_l$. Let D'' be a TRDS of the new path $uvx_1x_2 \ldots x_l st$. By Proposition 1, $\{u, v, s, t\} \subseteq D''$. Hence, it can be seen that $D := D' \cup D'' \backslash \{u, v, s, t\}$ is a TRDS of G. Therefore, by Proposition 2, we have

$$\begin{aligned} |D \backslash D'| &\leq \gamma_{tr}(P_{l+4}) - 4 = l + 4 - 2\left\lfloor \frac{l+4-2}{4} \right\rfloor - 4 \\ &\leq l - 2\left(\frac{l+2}{4} - 1\right) = \frac{l}{2} + 1. \end{aligned}$$

Case 2 At least one of the vertices x and y is not in D' and $l \not\equiv 3 \pmod 4$. If $x, y \notin D'$, then we add a TRDS of $P \backslash \{x_1, x_l\}$ to D'. If $x \in D'$ and $y \notin D'$, then we add a TRDS of $P \backslash \{x_1, x_2\}$ to D'. In both cases, we have added at most

$$\gamma_{tr}(P_{l-2}) = l - 2 - 2\left\lfloor \frac{l-2-2}{4} \right\rfloor \leq l - 2 - 2\left(\frac{l-4}{4} - \frac{2}{4}\right) = \frac{l}{2} + 1$$

vertices to D' to obtain a TRDS of G.

Case 3 At least one of the vertices x and y is not in D' and $l \equiv 3 \pmod 4$. In this case, let $x \notin D'$. So we add a TRDS of $P\backslash\{x_1\}$ to D' and we have

$$\gamma_{tr}(P_{l-1}) = l - 1 - 2\left\lfloor \frac{l-1-2}{4} \right\rfloor = l - 1 - 2\left(\frac{l-3}{4}\right) = \frac{l}{2} + \frac{1}{2}.$$

The result thus follows. □

Let G be a graph of order n with $\delta \geq 2$, and D be a subset of V. A vertex of degree greater than two is called a *large vertex*. We denote the set of large vertices in G by $L(G)$ and the set of vertices of degree two by $S(G)$. If there is no confusion, then we denote these two sets by L and S, respectively.We call a vertex v a *bad vertex* with respect to D if it has no neighbor in D, or it is an isolated vertex in $\langle V \setminus D \rangle$. Otherwise, we call v a *good vertex* with respect to D. It is obvious that D is a TRDS of G if and only if G has no bad vertex with respect to D.

3 Proof of the Main Result

We are now ready to prove our main result.

Proof of Theorem 3 The proof is by induction on n. For $n \leq 32$, if $\delta \leq n-2$, then by Theorem 1, $\gamma_{tr}(G) \leq n - \delta$; and if $\delta = n-1$, then G is a complete graph. Thus, in both cases, as $\delta \geq 2$, we have $\gamma_{tr}(G) \leq n - 2 \leq n - \sqrt[3]{\frac{n}{4}}$.

Now assume that $n > 32$, $\delta \geq 2$, and the statement is true for all graphs of order less than n. Recall that, an edge e is called a *bridge* if after removing it the number of components of the graph is increased.

Claim 1 *If G has a bridge incident with two large vertices, then $\gamma_{tr}(G) \leq n - \sqrt[3]{\frac{n}{4}}$.*

Proof Let $e = uv$ be a bridge in G, where u and v are large vertices. Let G_1, G_2 be the two components of $G\backslash e$, containing u and v, respectively. If $n_1 := n(G_1)$ and $n_2 := n(G_2)$ are more than three, then by the induction hypothesis,

$$\gamma_{tr}(G) \leq \gamma_{tr}(G_1) + \gamma_{tr}(G_2) \leq n_1 - \sqrt[3]{\frac{n_1}{4}} + n_2 - \sqrt[3]{\frac{n_2}{4}} \leq n - \sqrt[3]{\frac{n_1+n_2}{4}}$$
$$= n - \sqrt[3]{\frac{n}{4}}.$$

Otherwise, let $n_1 = 3$. Then $n_2 \geq 4$ and $\delta(G_2) \geq 2$. By the induction hypothesis, $\gamma_{tr}(G_2) \leq n_2 - \sqrt[3]{\frac{n_2}{4}}$. Moreover, if D is a TRDS of G_2, then either $v \in D$ or $v \notin D$. In either case, the set $D' := D \cup \{u\}$ or $D' := D \cup V(G_1)\backslash\{u\}$ is a TRDS of G, respectively. Hence, we have

$$\gamma_{tr}(G) \leq |D'| \leq 2 + \gamma_{tr}(G_2) \leq 3 - \sqrt[3]{\frac{3}{4}} + n_2 - \sqrt[3]{\frac{n_2}{4}} \leq n - \sqrt[3]{\frac{3+n_2}{4}} = n - \sqrt[3]{\frac{n}{4}}.$$

□

Let e be an edge in G incident with two large vertices, and $G' = G \backslash e$. If G' is disconnected, then by Claim 1, we are done. In the case that G' is connected, since $\gamma_{tr}(G) \le \gamma_{tr}(G')$, it is enough to find an upper bound for $\gamma_{tr}(G')$. Therefore, we can delete all the edges incident with two large vertices, and assume that $L(G)$ is independent. Note that if $L(G) = \emptyset$, then G is a cycle, and by Proposition 2(ii), the statement is true. Thus, we further assume that $L(G) \neq \emptyset$. Recall that S is the set of vertices of degree two in G.

Claim 2 *If the set of edges in $\langle S \rangle$ is not a matching, then $\gamma_{tr}(G) \le n - \sqrt[3]{\frac{n}{4}}$.*

Proof It is obvious that every component of $\langle S \rangle$ is a path. For a contradiction, let $P = x_1 x_2 \ldots x_l$, $l \ge 3$, be a component of $\langle S \rangle$ and vertices x and y be the neighbors of x_1 and x_l in $G \backslash P$, respectively. So $x, y \in L$.

Case 1 $x = y$.
If $G \backslash P$ is of order at least four, then by the induction hypothesis, it has a TRDS, say D', of order at most $n - l - \sqrt[3]{\frac{n-l}{4}}$; otherwise, let $D' := \{x\}$. Thus, in both cases, $|D'| \le n - l - \sqrt[3]{\frac{n-l}{4}}$. Moreover, for $C_{l+1} := \langle P \cup \{x\} \rangle$, there is a $\gamma_{tr}(C_{l+1})$-set which contains x and also a $\gamma_{tr}(C_{l+1})$-set which does not contain x, and $\gamma_{tr}(C_{l+1}) \le l - \sqrt[3]{\frac{l}{4}}$. Therefore, we can extend D' depending on $x \in D'$ or $x \notin D'$ to a TRDS of G with at most $n - l - \sqrt[3]{\frac{n-l}{4}} + l - \sqrt[3]{\frac{l}{4}} \le n - \sqrt[3]{\frac{n}{4}}$ vertices.

Case 2 $x \neq y$ and $G \backslash P$ is connected.
Since $x \neq y$, we have $\delta(G \backslash P) \ge 2$ and $n(G \backslash P) \ge 4$. Thus, by the induction hypothesis, we have $\gamma_{tr}(G \backslash P) \le n - l - \sqrt[3]{\frac{n-l}{4}}$. Hence, by Lemma 1,

$$\gamma_{tr}(G) \le \gamma_{tr}(G \backslash P) + \frac{l}{2} + 1 \le n - l - \sqrt[3]{\frac{n-l}{4}} + \frac{l}{2} + 1. \tag{*}$$

Let $f(l) = \sqrt[3]{\frac{n-l}{4}} + \frac{l}{2} - 1$. Since $f'(l) = \frac{-1}{12}(\frac{n-l}{4})^{\frac{-2}{3}} + \frac{1}{2} > 0$, $f(l)$ is an increasing function. Therefore, for $l \ge 3$, since $n \ge 32$, we have:

$$f(l) \ge f(3) = \sqrt[3]{\frac{n-3}{4}} + \frac{3}{2} - 1 = \sqrt[3]{\frac{n-3}{4}} + \frac{1}{2} \ge \sqrt[3]{\frac{n}{4}}.$$

Hence, $\gamma_{tr}(G) \le n - f(l) \le n - \sqrt[3]{\frac{n}{4}}$.

Case 3 $x \neq y$ and $G \backslash P$ is disconnected.
If each component of $G \backslash P$ is of order at least 4, then by the induction hypothesis for every component, we have $\gamma_{tr}(G \backslash P) \le n - l - \sqrt[3]{\frac{n-l}{4}}$. Hence, by Lemma 1, we get again the inequality (*), and the desired result follows likewise.

Now, without loss of generality, suppose that the component which contains x, say G_x, is of order three. Let $G' := \langle V(G_x) \cup P \rangle$ and $l' := n(G')$. If $n(G \backslash G') = 3$, then every TRDS of $\langle P \cup \{x, y\} \rangle$ is a TRDS of G. Thus,

$$\gamma_{tr}(G) \le \gamma_{tr}(P_{n-4}) \le \frac{n}{2} + 1 \le n - \sqrt[3]{\frac{n}{4}},$$

and we are done. So assume that $n(G \backslash G') > 3$.

By the induction hypothesis, $\gamma_{tr}(G \backslash G') \le n - l' - \sqrt[3]{\frac{n-l'}{4}}$. On the other hand, the union of a TRDS of $G \backslash G'$ and a TRDS of $\langle P \cup \{x\} \rangle$ is a TRDS of G. Thus, as $\gamma_{tr}(P_{l'-2}) = l' - 2 - 2\lfloor \frac{l'-2-2}{4} \rfloor \le \frac{l'}{2} + 2$, we have

$$\begin{aligned} \gamma_{tr}(G) &\le \gamma_{tr}(G \backslash G') + \gamma_{tr}(P_{l'-2}) \\ &\le n - l' - \sqrt[3]{\frac{n-l'}{4}} + \frac{l'}{2} + 2 \\ &= n - \left(\sqrt[3]{\frac{n-l'}{4}} + \frac{l'}{2} - 2 \right). \end{aligned}$$

Now, consider $f(l') = \sqrt[3]{\frac{n-l'}{4}} + \frac{l'}{2} - 2$. Similar to Case 2, $f(l')$ is an increasing function and for $l' \ge 6$, we have

$$\gamma_{tr}(G) \le n - \left(\sqrt[3]{\frac{n-6}{4}} + \frac{6}{2} - 2 \right) \le n - \sqrt[3]{\frac{n}{4}}.$$

□

From now on, we assume that $\langle S \rangle$ is a matching (note that, $\langle S \rangle$ can also contains isolated vertices).

A set B of vertices in G such that $N_G(x) \cap N_G(y) = \emptyset$ for all $x, y \in B$ is called an *open packing*.

Claim 3 *If the set L contains no open packing of size at most* $\sqrt[3]{\frac{n}{4}}$, *then* $\gamma_{tr}(G) \le n - \sqrt[3]{\frac{n}{4}}$.

Proof Suppose $\Delta(G) = n - 1$. Let x be a vertex with maximum degree and y be a neighbor of x with minimum degree. If $\{x, y\}$ is not a TRDS of G, then there is a vertex z in G such that $N_G(z) = \{x, y\}$; so $\deg_G(z) = 2$, and thus $\deg_G(y) = 2$. Since x is adjacent to all vertices, it is easy to see that the set $\{x, y, z\}$ is a TRDS of G. Hence, in this case, $\gamma_{tr}(G) \le n - \sqrt[3]{\frac{n}{4}}$.

Now assume that $\Delta(G) \le n - 2$. Let $a = |S|$, $b = |L|$ and $k = \sqrt[3]{\frac{n}{4}}$. By Theorem 2, $\gamma_{tr}(G) \le n - \frac{\Delta}{2} - 1$. Suppose on the contrary that $\gamma_{tr}(G) > n - k$. Then

$$n - k < n - \frac{\Delta}{2} - 1,$$

which implies that

$$\Delta < 2k - 2. \tag{1}$$

On the other hand, by Claim 2, every vertex in S has a neighbor in L. Let p be the number of edges between S and L. It follows that $a \leq p \leq b\Delta$. Hence, since $a + b = n$, we have

$$n - b \leq b\Delta,$$

and thus

$$\frac{n}{\Delta + 1} \leq b. \tag{2}$$

In what follows, we shall use the probabilistic method and the above inequalities to show that there exists an open packing of size k in L which thus leads to a contradiction. For this purpose, let $<$ be a uniformly chosen total ordering of L. Define

$$I := \{v : \; v, w \in L \text{ have a common neighbor } \Rightarrow v < w\}.$$

In fact, I is a maximal open packing which contains the least vertex of L with respect to the order $<$. Let X_v be the indicator random variable for $v \in I$ and $X := \sum_{v \in V} X_v = |I|$. For each $v \in L$, since the degree of each vertex in $N_G(v)$ is two, there are at most Δ vertices of distance two from v. Hence, a vertex $v \in L$ is in I when v is the least vertex with respect to $<$ among the set of vertices of distance two from v together with $\{v\}$. Therefore, for every $v \in L$,

$$E(X_v) = Pr(v \in I) \geq \frac{1}{\Delta + 1}.$$

Now, by linearity of expectation function and (2),

$$E(X) \geq \sum_{v \in L} \frac{1}{\Delta + 1} = \frac{b}{\Delta + 1} \geq \frac{n}{(\Delta + 1)^2}.$$

Thus, by (1),

$$E(X) \geq \frac{n}{(2k-1)^2} \geq \frac{n}{4k^2} \geq k.$$

Hence, there exists a specific ordering $<$ on L with $|I| \geq k$. □

From now on, we assume that L contains an open packing of size $k \geq \sqrt[3]{\frac{n}{4}}$. Let $X = \{x_1, x_2, \ldots, x_k\}$ be an open packing of G in L. If for some i, $1 \leq i \leq k$, the induced subgraph $G' := \langle N_G(x_i) \rangle$ has no isolated vertex, then since G is connected and L is an independent set, $V(G) = N_G[x_i]$ and $E(G')$ is a matching. Hence, the set consisting of vertex x_i and two adjacent vertices in G' is a $\gamma_{tr}(G)$-set of size 3.

Thus, $\gamma_{tr}(G) \leq 3 \leq n-k$. Otherwise, for every i, $1 \leq i \leq k$, let y_i be an isolated vertex in $\langle N_G(x_i)\rangle$, and $Y := \{y_1, y_2, \ldots, y_k\}$. Note that since X is an open packing, the vertices y_i, $1 \leq i \leq k$, are distinct. We shall now construct a set D_i^c, recursively on i, and let $D_i = V(G)\backslash D_i^c$. In step i, denote the set of bad vertices with respect to D_i by Z_i.

For $i = 0$, let D_0^c be the set obtained from $X \cup Y$ by deleting from X the neighbors of adjacent vertices in Y. Note that the degree of each vertex in $\langle D_0^c\rangle$ is one and also a vertex is a bad vertex with respect to D_0 if and only if it is an isolated vertex in $\langle D_0\rangle$. We denote the bad vertices with respect to D_0 in S by $z_1, z_2, \ldots, z_t$ and the bad vertices with respect to D_0 in L by $z_{t+1}, \ldots, z_s$.

We construct D_i^c recursively with the following properties:

(1) The degree of every vertex of $(S \cup X) \cap D_i^c$ in $\left\langle D_i^c\right\rangle$ is equal to one.
(2) For each $x_j \in X \cap D_i^c$, $N(x_j) \subseteq \{y_j\}$.
(3) For $i \geq 1$, $D_i^c \subseteq D_{i-1}^c \cup \{z_i\}$.
(4) For $i \geq 1$, $Z_i \subseteq Z_{i-1}\backslash\{z_i\}$.

Assume that D_{i-1}^c is constructed for $0 \leq i-1 \leq s-1$ with the above properties. If $1 \leq i \leq t$, then we construct D_i^c as follows. Note that since for i, $1 \leq i \leq t$, $z_i \in S$, by Property (3), $D_{i-1}^c \subseteq S \cup X$. Hence, for $1 \leq i \leq t$, Property (1) is equivalent to that the degree of every vertex in D_{i-1}^c in $\left\langle D_{i-1}^c\right\rangle$ is one. If z_i is a good vertex with respect to D_{i-1}, then let $D_i^c := D_{i-1}^c$; otherwise, by Property (1), z_i is an isolated vertex in $\langle D_{i-1}\rangle$. Since $z_i \in Z_0$ (i.e., z_i is a bad vertex with respect to D_0), we have $N_G(z_i) \subseteq D_0^c \subseteq X \cup Y$. Hence, by Claim 2 and since X is an open packing, z_i is adjacent to some vertices x_{a_i} and y_{b_i}. In this case, let $D_i^c := D_{i-1}^c \cup \{z_i\} - \{x_{a_i}, x_{b_i}, y_{a_i}\}$. Properties (2) and (3) are clearly satisfied. Since in $\langle D_i^c\rangle$ the degrees of z_i and y_{b_i} are one and the degrees of the other vertices of D_i^c have not changed, the degree of each vertex in $\langle D_i^c\rangle$ is one (Property (1)). Hence, if in this step a vertex is a bad vertex with respect to D_i, then it is an isolated vertex in $\langle D_i\rangle$. Moreover, if a vertex is a bad vertex with respect to D_i and not in Z_{i-1}, then it is in $N_G[\{z_i, x_{a_i}, x_{b_i}, y_{a_i}\}]$. But the only neighbor of z_i in D_i is x_{a_i} which is adjacent to $y_{a_i} \in D_i$. Hence, the neighbors of z_i are not bad vertices with respect to D_i. Since the vertices $\{x_{a_i}, y_{a_i}, x_{b_i}\}$ are added to D_i and they have already dominated by D_{i-1}, these vertices and their neighbors are not isolated vertices in $\langle D_i\rangle$. Therefore, in this process we don't create new bad vertices with respect to D_i. Moreover, z_i is not a bad vertex with respect to D_i. Hence, $Z_i \subseteq Z_{i-1}\backslash\{z_i\}$ (Property (4)).

If $t+1 \leq i \leq s$, then we construct D_i^c as follows. If z_i is a good vertex with respect to D_{i-1}, then set $D_i^c := D_{i-1}^c$; otherwise, proceed as follows. Since $z_i \in Z_0$ (i.e., z_i is a bad vertex with respect to D_0), $N_G(z_i) \subseteq X \cup Y$. Moreover, $z_i \in L$. Thus, the neighbors of z_i are in $Y \cap D_i^c$, say $y_{t_1}, \ldots, y_{t_r}$. Let $D_i^c := D_{i-1}^c \cup \{z_i\}\backslash\{x_{t_1}, \ldots, x_{t_r}, y_{t_r}\}$. Properties (2) and (3) are clearly satisfied. By Properties (1), (2) and the above construction, the degree of every vertex of $(S \cup X) \cap D_i^c$ in $\left\langle D_i^c\right\rangle$ is one (Property (1)). Thus, vertices of set $(S \cup X) \cap D_i^c$ are good vertices with respect to D_i. On the other hand, if a vertex is a bad vertex with respect to D_i and is not in Z_{i-1}, then it is in $N_G[\{x_{t_1}, \ldots, x_{t_r}, y_{t_r}, z_i\}]$. For x_{t_h}, $1 \leq h < r$, since $x_{t_h} \in L$ and $x_{t_h} \notin Z_{i-1}$, x_{t_h} is dominated by $D_{i-1}\backslash L (\subseteq D_i)$. Also, since we add x_{t_h} to D_i and the only neighbor of x_{t_h} in D_i^c is y_{t_h} (by Properties (1) and (2)) which is adjacent to $z_i (\in D_i^c)$, there is no bad vertex with respect to D_i in $N_G[x_{t_h}]$. Also, note that z_i is dominated by $y_{t_r} (\in D_i)$ and it is adjacent to $y_{t_1} (\in D_i^c)$. Moreover, $N_G(z_i) \cap D_i = \{y_{t_r}\}$ and y_{t_r} is adjacent to $x_{t_r} \in D_i$. Hence, there is no bad vertex in $N_G[z_i]$ with respect to D_i. Also, for y_{t_r} and x_{t_r} we simply observe that there is no bad vertex in $N_G[y_{t_r}]$ and $N_G[x_{t_r}]$ with respect to D_i. Thus, the set of bad vertices in G with respect to D_i is a subset of $Z_{i-1} \backslash \{z_i\}$ (Property (4)).

Therefore, in this process the number of bad vertices is decreased until we have no bad vertices. Moreover, in each step corresponding to deleting a vertex y_i from D_{i-1}^c, we add one vertex of $V(G) \backslash X$ to D_{i-1}^c, but we only delete vertices of X. Therefore, following this process, we end up with the set D_s^c of size at least $|Y| = k$ such that there is no bad vertex with respect to D_s. Hence, D_s is a TRDS of G of size at most $n - k$. The proof is now complete. □

Remark For integer $r \geq 2$, let G_r be a bipartite graph formed by taking as one partite set a set $A = \{1, \ldots, r\}$, and as the other partite set B all the 2-element subsets of A, and joining each element of A to those subsets it is a member of. Note that $n = n(G_r) = r + \binom{r}{2}$. Every TRDS D of G_r contains at least $r - 1$ vertices of A (for every vertex in B to have a neighbor in D) and therefore every 2-subset of these $r - 1$ elements of A have to be in D; i.e., at least $\binom{r-1}{2}$ vertices of B have to be in D (unless $D = V(G)$). Also, to dominate the rth vertex of A, say a, a vertex $\{x, a\}$, $x \in A - \{a\}$, of B have to be in D. Thus, every TRDS of G_r contains at least $r + \binom{r-1}{2}$ vertices. On the other hand, the union of set $S = \{1, \ldots, r - 1\}$ and all 2-element subsets of S and vertex $\{1, r\}$ is a TRDS of G_r of size $r + \binom{r-1}{2}$. Therefore, $\gamma_{tr}(G_r) = r + \binom{r-1}{2}$. Hence, $n - \gamma_{tr}(G_r) = r - 1 = \theta(r) = \theta(\sqrt{n})$. Therefore, $\gamma_{tr}(G_r) = n - \theta(\sqrt{n})$.

We end this paper by proposing the following:

Conjecture *If G is a connected graph of order* n, $n \geq 4$, *and minimum degree* $\delta \geq 2$, *then*

$$\gamma_{tr}(G) \leq n - \theta(\sqrt{n}).$$

References

1. Cockayne, E.J., Dawes, R.M., Hedetniemi, S.T.: Total domination in graphs. Networks **10**(3), 211–219 (1980)
2. Cyman, J., Raczek, J.: On the total restrained domination number of a graph. Australas. J. Combin. **36**, 91–100 (2006)
3. Gravier, S.: Total domination number of grid graphs. Discrete Appl. Math. **121**(1–3), 119–128 (2002)
4. Haynes, T.W., Hedetniemi, S.T., Slater, P.J. (eds.): Domination in Graphs: Advanced Topics. Marcel Dekker, New York (1998)
5. Haynes, T.W., Hedetniemi, S.T., Slater, P.J. (eds.): Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998)
6. Henning, M.A.: A survey of selected recent results on total domination in graphs. Discrete Math. **309**(1), 32–63 (2009)
7. Henning, M.A., Maritz, J.E.: Total restrained domination in graphs with minimum degree two. Discrete Math. **308**(10), 1909–1920 (2008)
8. Koh, K.M., Maleki, Z., Omoomi, B.: Max-min total restrained domination number. Ars. Combin. (to appear)

9. Koh, K.M., Maleki, Z., Omoomi, B.: On the total restrained domination edge critical graphs. Ars. Combin. (to appear)
10. Ma, D.-X., Chen, X.-G., Sun, L.: On total restrained domination in graphs. Czechoslov. Math. J. **55**(1), 165–173 (2005)
11. Shan, E., Kang, L., Henning, M.A.: A characterization of trees with equal total domination and paired-domination numbers. Australas. J. Combin. **30**, 31–39 (2004)
12. Telle, J.A., Proskurowski, A.: Algorithms for vertex partitioning problems on partial k-trees. SIAM J. Discrete Math. **10**(4), 529–550 (1997)
13. West, D.B.: Introduction to Graph Theory, 2nd edn. Prentice Hall, Upper Saddle River (2001)