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Chromatic equivalence classes of certain generalized polygon trees, III☆

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Abstract

Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, if $P(G)=P(H)$. A set of graphs $\mathscr{S}$ is called a *chromatic equivalence class* if for any graph H that is chromatically equivalent with a graph G in $\mathscr{S}$, then $H \in \mathscr{S}$. Peng et al. (Discrete Math. 172 (1997) 103–114), studied the chromatic equivalence classes of certain generalized polygon trees. In this paper, we continue that study and present a solution to Problem 2 in Koh and Teo (Discrete Math. 172 (1997) 59–78).

MSC: primary 05C15

1. Introduction

The graphs that we consider are finite, undirected and simple. Let $P(G,\lambda)$ or simply $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are said to be *chromatically equivalent*, and we write $G \sim H$, if $P(G)=P(H)$. Trivially, the relation "$\sim$" is an equivalence relation on the class of graphs. A graph G is *chromatically unique* if G is isomorphic with H for any graph H such that $G \sim H$. A set of graphs $\mathscr{S}$ is called a *chromatic equivalence class* if for any graph H that is

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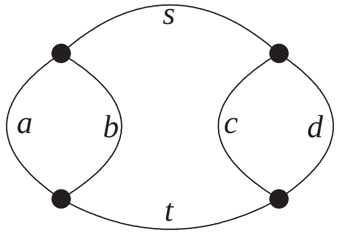

Fig. 1. $G_t^s(a,b;\ c,d)$.

chromatically equivalent with a graph G in $\mathscr{S}$, then $H \in \mathscr{S}$. Although chromatically unique graphs have been the subject of many recent papers (see [2,3]), relatively fewer results concerning the chromatically equivalence class of graphs are known.

A path in G is called a *simple* path if the degree of each interior vertex is two in G. A *generalized polygon tree* is a graph defined recursively as follows. A cycle C_p ($p \geqslant 3$) is a generalized polygon tree. Next, suppose H is a generalized polygon tree containing a simple path P_k, where $k \geqslant 1$. If G is a graph obtained from the union of H and a cycle C_r, where $r > k$, by identifying P_k in H with a path of length k in C_r, then G is also a generalized polygon tree. Consider the generalized polygon tree $G_t^s(a,b;\ c,d)$ with three interior regions shown in Fig. 1. The integers a, b, c, d, s and t represent the lengths of the respective paths between the vertices of degree three, where $s \geqslant 0$, $t \geqslant 0$. Without loss of generality, assume that $a \leqslant b$, and $a \leqslant c \leqslant d$. Thus, $\min\{a,b,c,d\} = a$. Let $r = s+t$. We now form a family $\mathscr{C}_r(a,b;\ c,d)$ of the graphs $G_t^s(a,b;\ c,d)$ where the values of a, b, c, d and r are fixed but the values of s and t vary; that is

$$\mathscr{C}_r(a,b;\ c,d) = \{G_t^s(a,b;\ c,d) \mid r = s+t, s \geqslant 0,\ t \geqslant 0\}.$$

It is clear that the families $\mathscr{C}_0(a,b;\ c,d)$ and $\mathscr{C}_1(a,b;\ c,d)$ are singletons.

Note that $G_t^s(a,b;\ c,d)$ is a connected $(n,n+2)$-graph, whose chromatic polynomials were computed by Chao and Zhao (see [1]), who also determined several chromatic equivalence classes, excluding among others the graph $G_t^s(a,b;\ c,d)$.

In [3], Koh and Teo posed the following problem.

Problem (Koh and Teo [**3**])**.** Study the chromaticity of $\mathscr{C}_r(a,b;\ c,d)$ in general.

In order to solve the problem above, Peng et al. in [6], showed that $\mathscr{C}_r(a,b;\ c,d)$ is a chromatic equivalence class for a, b, c, d at least $r+3$. In [4], we characterized the chromaticity of $\mathscr{C}_1(a,b;\ c,d)$. Also in [5], we characterized the chromaticity of $\mathscr{C}_r(a,b;\ c,d)$ for $r \geqslant 2$ and the minimum of $a,b,c,$ and d equals to $r+2$. In [8], Xu et al. solved the problem for $r=0$. In this paper, we present necessary and sufficient conditions for $\mathscr{C}_r(a,b;\ c,d)$ to be a chromatic equivalence class when $r \geqslant 2$ and the minimum of a, b, c and d less than $r+2$. Thus the problem above is solved completely.

2. Basic results

In this section, we give some known results that will be used to prove our main theorems. The first result lists some well-known necessary conditions for chromatic equivalence. The girth of G, denoted by $g(G)$, is the length of a shortest cycle of G.

Theorem A (Whitney [7]). *Let G and H be chromatically equivalent graphs. Then*

(a) $|V(G)| = |V(H)|$;
(b) $|E(G)| = |E(H)|$;
(c) $g(G) = g(H)$;
(d) *G and H have the same number of shortest cycles.*

The next known result gives the chromatic polynomial of $G_t^s(a,b;\ c,d)$. In [1], Chao and Zhao also determined the chromatic polynomial of this graph, but we shall use the computed chromatic polynomial of $G_t^s(a,b;\ c,d)$ in [6] to prove our main results.

Theorem B (Peng et al. [6]). *Let the order of $G_t^s(a,b;\ c,d)$ be n ($n = a+b+c+d+r-2$), and $x = 1-\lambda$. Then we have*

$$P(G_t^s(a,b;\ c,d)) = \frac{(-1)^n x}{(x-1)^2}\, Q(G_t^s(a,b;\ c,d)),$$

where

$$\begin{aligned} Q(G_t^s(a,b;\ c,d)) = {} & (x^{n+1} - x^{a+b+r} - x^{c+d+r} + x^{r+1} - x) \\ & - (1 + x + x^2) + (x+1)(x^a + x^b + x^c + x^d) \\ & - (x^{a+c} + x^{a+d} + x^{b+c} + x^{b+d}). \end{aligned}$$

The following theorem is a consequence of Theorem B and it implies that $P(G_r^0(a,b;\ c,d)) = P(G_t^s(a,b;\ c,d))$, where $r = s+t$.

Theorem C (Chao and Zhao [1], and Peng et al. [6]). *All the graphs in $\mathscr{C}_r(a,b;\ c,d)$ are chromatically equivalent.*

The next result follows from Lemma 2 in [6] and Case 1 in the proof of Theorem 6 in [6]. Note that despite the frequent mention of the condition $\min\{a,b,c,d\} \geqslant r+3$, it is not used in the proof of Case 1 in Theorem 6 in [6].

Theorem D (Peng et al. [6]). *If $G_t^s(a,b;\ c,d)$ and $G_{t'}^{s'}(a',b';\ c',d')$ are chromatically equivalent and $s+t = s'+t'$, then $G_{t'}^{s'}(a',b';\ c',d') \in \mathscr{C}_r(a,b;\ c,d)$, where $r = s+t$.*

In [6], Peng et al. present the following sufficient condition for $\mathscr{C}_r(a,b;\ c,d)$ to be a chromatic equivalence class.

Theorem E. *The family of graphs* $\mathscr{C}_r(a,b;\ c,d)$ *is a chromatic equivalence class if* $\min\{a,b,c,d\} \geqslant r+3$.

Xu et al. in [8] studied the chromaticity of $\mathscr{C}_r(a,b;\ c,d)$ for $\min\{a,b,c,d\} = 1$.

Theorem F (Xu et al. [8]). *The family of graphs*

$$\mathscr{C}_0(1,b;\ c,d) \cup \mathscr{C}_{b-1}(1,c;\ 1,d) \cup \mathscr{C}_{c-1}(1,b;\ 1,d) \cup \mathscr{C}_{d-1}(1,b;\ 1,c),$$

where $b,c,d \geqslant 2$, *is a chromatic equivalence class. Also the family of graphs*

$$\mathscr{F} = \mathscr{C}_r(1,b;\ c,d) \cup \mathscr{C}_{c-1}(1,b;\ r+1,d) \cup \mathscr{C}_{d-1}(1,b;\ c,r+1),$$

where $r \geqslant 1$ *and* $b,c,d \geqslant 2$, *is a chromatic equivalence class except for* $r=2$ *and* $b=d=c+1$. *Moreover, for* $r=2$ *and* $b=d=c+1$ *the family of graphs*

$$\mathscr{C}_0(2,c;\ c+1,c+2) \cup \mathscr{C}_2(1,c+1;\ c,c+1) \cup \mathscr{C}_{c-1}(1,c+1;\ 3,c+1) \cup \mathscr{C}_c(1,c+1;\ c,3)$$

is a chromatic equivalence class.

Remark 1. In the family of graphs

$$\mathscr{F} = \mathscr{C}_r(1,b;\ c,d) \cup \mathscr{C}_{c-1}(1,b;\ r+1,d) \cup \mathscr{C}_{d-1}(1,b;\ c,r+1),$$

if $c=d=r+1$, then $\mathscr{F} = \mathscr{C}_r(1,b;\ r+1,r+1)$. Therefore by Theorem F, $\mathscr{C}_r(1,b;\ r+1,r+1)$ is a chromatic equivalence class.

In [4], Omoomi and Peng gave necessary and sufficient conditions for $\mathscr{C}_r(a,b;\ c,d)$ to be a chromatic equivalence class when $r=1$. As a consequence, they obtained all the families of chromatic equivalence classes containing $\mathscr{C}_1(a,b;c,d)$ which is not chromatic equivalence class, where $\min\{a,b,c,d\} \geqslant 2$. We list them in the following theorem.

Theorem G. *Each of the following families is a chromatic equivalence class*:

(a) $\mathscr{C}_1(2,3;\ 3,5) \cup \mathscr{C}_3(2,3;\ 2,4)$;
(b) $\mathscr{C}_1(3,5;\ 5,8) \cup \mathscr{C}_5(2,6;\ 4,5)$;
(c) $\mathscr{C}_1(3,c;\ c+1,c+3) \cup \mathscr{C}_3(2,c+1;\ c,c+2)$, *for any* $c \geqslant 3$;
(d) $\mathscr{C}_1(3,c+3;\ c,c+1) \cup \mathscr{C}_3(2,c+2;\ c,c+1)$, *for any* $c \geqslant 3$;
(e) $\mathscr{C}_1(3,3;\ c,c+2) \cup \mathscr{C}_{c-1}(2,4;\ 3,c+1)$, *for any* $c \geqslant 3$;
(f) $\mathscr{C}_1(3,b;\ 3,b+2) \cup \mathscr{C}_{b-1}(2,b+1;\ 3,4)$, *for any* $b \geqslant 3$.

Remark 2. If $c=2$ in the families (c) and (d), then we get the family (a).

The next known result gives necessary and sufficient conditions for $\mathscr{C}_r(a,b;\ c,d)$ to be a chromatic equivalence class when $r \geqslant 2$ and $\min\{a,b,c,d\} = r+2$.

Theorem H (Omoomi and Peng [5]). *The family of graphs* $\mathscr{C}_r(a,b;\ c,d)$ *is a chromatic equivalence class if* $r \geqslant 2$ *and* $\min\{a,b,c,d\} = r+2$, *except the two families* $\mathscr{C}_r(r+2,b;\ b+1,b+r+2)$ *and* $\mathscr{C}_r(r+2,c+r+2;\ c,c+1)$.

The following corollary follows from Theorem H.

Corollary. *The following two families of graphs are chromatic equivalence classes.*

(a) $\mathscr{C}_r(r+2,b;\ b+1,b+r+2)\cup\mathscr{C}_{r+2}(r+1,b+1;\ b,b+r+1)$, *for* $b\geqslant r+2\geqslant 2$;
(b) $\mathscr{C}_r(r+2,c+r+2;\ c,c+1)\cup\mathscr{C}_{r+2}(r+1,c+r+1;\ c,c+1)$, *for* $c\geqslant r+2\geqslant 2$.

Proof. From the proof of Theorem H, we get the chromatic equivalence classes (a) and (b) for $r\geqslant 2$. If $r=0$, then we have

(a) $\mathscr{C}_0(2,b;\ b+1,b+2)\cup\mathscr{C}_2(1,b+1;\ b,b+1)$, for $b\geqslant 2$;
(b) $\mathscr{C}_0(2,c+2;\ c,c+1)\cup\mathscr{C}_2(1,c+1;\ c,c+1)$, for $c\geqslant 2$;

which are also chromatic equivalence classes. This follows from the proof of Theorem 1 in [8]. If $r=1$, then we have

(a) $\mathscr{C}_1(3,b;\ b+1,b+3)\cup\mathscr{C}_3(2,b+1;\ b,b+2)$, for $b\geqslant 3$;
(b) $\mathscr{C}_1(3,c+3;\ c,c+1)\cup\mathscr{C}_3(2,c+2;\ c,c+1)$, for $c\geqslant 3$;

which are exactly the families of graphs in (c) and (d) of Theorem G. □

Remark 3. The families of graphs in Corollary of Theorem H can be written as follows.

(a) $\mathscr{C}_r(a,r+2;\ a+1,a+r+2)$, for $r\geqslant 0,\ a\geqslant 2$;
(b) $\mathscr{C}_r(a,a+1;\ r+2,a+r+2)$, for $r\geqslant 0,\ a\geqslant 2$;
(c) $\mathscr{C}_r(r-1,c+1;\ c,c+r-1)$, for $r\geqslant 2,\ c\geqslant r$;
(d) $\mathscr{C}_r(r-1,c+r-1;\ c,c+1)$, for $r\geqslant 2,\ c\geqslant r$.

3. Main theorems

Suppose that H is a graph such that $P(H)=P(G_t^s(a,b;c,d))$. Then by Lemma 4 and Theorem 2 in [1], we know that $H=G_{t'}^{s'}(a',b';c',d')$, where $a',b',c',d'\geqslant 1$. The question now is whether or not the graph $G_{t'}^{s'}(a',b';c',d')$ is in the family $\mathscr{C}_r(a,b;\ c,d)$. In other words, is $\mathscr{C}_r(a,b;\ c,d)$ a chromatic equivalence class? In this section, we shall present necessary and sufficient conditions for $\mathscr{C}_r(a,b;\ c,d)$ to be a chromatic equivalence class.

Theorem 1. *The family of graphs* $\mathscr{C}_r(a,b;\ c,d)$ *is not a chromatic equivalence class for* $r\geqslant 2$ *and* $2\leqslant\min\{a,b,c,d\}\leqslant r+1$, *if and only if it is one of the following nine families*:

(a) $\mathscr{C}_5(2,6;\ 4,5)$;
(b) $\mathscr{C}_3(2,c+1;\ c,c+2)$, *for any* $c\geqslant 2$;
(c) $\mathscr{C}_3(2,c+2;\ c,c+1)$, *for any* $c\geqslant 2$;

(d) $\mathscr{C}_r(2,4;\; 3,r+2)$;
(e) $\mathscr{C}_r(2,r+2;\; 3,4)$;
(f) $\mathscr{C}_r(a,r+2;\; a+1,a+r+2)$, *for any* $a \geqslant 2$;
(g) $\mathscr{C}_r(a,a+1;\; r+2,a+r+2)$, *for any* $a \geqslant 2$;
(h) $\mathscr{C}_r(r-1,c+1;\; c,c+r-1)$, *for any* $c \geqslant r$;
(i) $\mathscr{C}_r(r-1,c+r-1;\; c,c+1)$, *for any* $c \geqslant r$.

Proof. The necessity follows immediately from Theorem G, Corollary of Theorem H, and Remark 3. To prove the sufficiency, we show that if $\mathscr{C}_r(a,b;\; c,d)$ is not a chromatic equivalence class for $r \geqslant 2$ and $2 \leqslant \min\{a,b,c,d\} \leqslant r+1$, then $\mathscr{C}_r(a,b;\; c,d)$ is one of the nine families of graphs.

Let $r \geqslant 2$ and $2 \leqslant \min\{a,b,c,d\} \leqslant r+1$. Suppose that $\mathscr{C}_r(a,b;\; c,d)$ is not a chromatic equivalence class. Let $G = G_t^s(a,b;\; c,d) \in \mathscr{C}_r(a,b;\; c,d)$ and $H \sim G$. By Lemma 4 and Theorem 2 in [1], $H = G_{t'}^{s'}(a',b';\; c',d')$, where $a',b',c',d' \geqslant 1$. Let $r' = s'+t'$. So $H \in \mathscr{C}_{r'}(a',b';\; c',d')$. Without loss of generality, we assume that $a \leqslant b$ and $a \leqslant c \leqslant d$; also $a' \leqslant b'$ and $a' \leqslant c' \leqslant d'$. We will now find G and H such that $H \notin \mathscr{C}_r(a,b;\; c,d)$. In other words, we will find a, b, c, d, and r; also a', b', c', d', and r' such that $H = G_{t'}^{s'}(a',b';\; c',d') \notin \mathscr{C}_r(a,b;\; c,d)$, and the answers will give us the nine families of graphs.

By Theorems A and B, we have $a+b+c+d+r = a'+b'+c'+d'+r'$, and $Q(G) = Q(H)$. Now we solve the equation $Q(G) = Q(H)$. After cancelling the terms x^{n+1}, $-x$ and $-(1+x+x^2)$, we have $Q_1(G) = Q_1(H)$, where

$$\begin{aligned} Q_1(G) &= x^{r+1} + (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} \\ &\quad - x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d}, \\ Q_1(H) &= x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r'+a'+b'} \\ &\quad - x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'}, \end{aligned}$$

$$a+b+c+d+r = a'+b'+c'+d'+r';$$

$$2 \leqslant a \leqslant r+1,\; a \leqslant b,\; a \leqslant c \leqslant d;\; a' \leqslant b',\; \text{and}\; a' \leqslant c' \leqslant d'.$$

Claim. $\min\{r+1,a,b,c,d\} = \min\{r'+1,a',b',c',d'\}$.

To show this claim, let $\min\{r+1,a,b,c,d\} = \alpha$ and $\min\{r'+1,a',b',c',d'\} = \beta$. Note that x^α in $Q_1(G)$ cannot be cancelled by any negative term of $Q_1(G)$, and similarly x^β in $Q_1(H)$ cannot be cancelled by any negative term of $Q_1(H)$. If $\alpha > \beta$, then x^β appears in $Q_1(H)$ but not in $Q_1(G)$, which is impossible. Similarly, if $\alpha < \beta$, then we have x^α in $Q_1(G)$ but not in $Q_1(H)$, and this is also impossible. Thus, we must have $\alpha = \beta$ as claimed.

Since $\min\{r+1,a,b,c,d\} = a \geqslant 2$, from the claim above, we have $r' \geqslant 1$ and $\min\{a',b',c',d'\} \geqslant 2$. If $r' = 1$, then from Theorem G, we get the first five families. If $r' \geqslant 2$ and $r' = r$, then by Theorem D, $H \in \mathscr{C}_r(a,b;\; c,d)$. Therefore, we may assume $r' \neq r$ when $r' \geqslant 2$.

Now let $r' \geqslant 2$ and let us look at the value of $\min\{a', b', c', d'\}$. If $\min\{a', b', c', d'\} \geqslant r' + 3$, then by Theorem E, the family $\mathscr{C}_{r'}(a', b'; c', d')$ is a chromatic equivalence class. Since $H \sim G$ and $H \in \mathscr{C}_{r'}(a', b'; c', d')$, we have $G \in \mathscr{C}_{r'}(a', b'; c', d')$. Thus $\mathscr{C}_{r'}(a', b'; c', d') = \mathscr{C}_r(a, b;\ c, d)$, that is $r' = r$. Therefore, we only need to consider $\min\{a', b', c', d'\} \leqslant r' + 2$.

If $\min\{a', b', c', d'\} = r' + 2$, then by Corollary of Theorem H, we have

$$H = G^0_{r'}(r'+2, b';\ b'+1, b'+r'+2) \sim G^0_{r'+2}(r'+1, b'+1;\ b', b'+r'+1) = G$$

or

$$H = G^0_{r'}(r'+2, c'+r'+2;\ c', c'+1) \sim G^0_{r'+2}(r'+1, c'+r'+1;\ c', c'+1) = G$$

for any $b', c' \geqslant r'+2$. Therefore, $H \notin \mathscr{C}_{r'+2}(r'+1, b'+1;\ b', b'+r'+1)$ or $H \notin \mathscr{C}_{r'+2}(r'+1, c'+r'+1;\ c', c'+1)$, for $b', c' \geqslant r'+2$. Note that $\mathscr{C}_{r'+2}(r'+1, b'+1;\ b', b'+r'+1)$ and $\mathscr{C}_{r'+2}(r'+1, c'+r'+1;\ c', c'+1)$ can be written as $\mathscr{C}_r(r-1, c+1;\ c, c+r-1)$, for $c \geqslant r$ and $\mathscr{C}_r(r-1, c+r-1;\ c, c+1)$, for $c \geqslant r$, respectively, which are the families (h) and (i).

We now need to consider $2 \leqslant \min\{a', b', c', d'\} \leqslant r'+1$. Since $2 \leqslant \min\{a, b, c, d\} \leqslant r+1$, $2 \leqslant \min\{a', b', c', d'\} \leqslant r'+1$, $r \neq r'$, and the chromatic equivalence is a symmetric relation, without loss of generality we may assume $r < r'$.

Since $\min\{r+1, a, b, c, d\} = a$ and $\min\{r'+1, a', b', c', d'\} = a'$, by the claim above, we have $a = a'$. Now, we have $Q_2(G) = Q_2(H)$, where

$$Q_2(G) = x^{r+1} + (x+1)(x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_2(H) = x^{r'+1} + (x+1)(x^{b'} + x^{c'} + x^{d'}) - x^{r'+a+b'} - x^{r'+c'+d'} - x^{a+c'} - x^{a+d'} - x^{b'+c'} - x^{b'+d'},$$

$$b + c + d + r = b' + c' + d' + r';$$

$$2 \leqslant a \leqslant r+1,\ a \leqslant b,\ a \leqslant c \leqslant d,\ a \leqslant b',\ a \leqslant c' \leqslant d',\ \text{and } r < r'.$$

We have either $b = b'$ or $b \neq b'$. If $b \neq b'$, we consider either $b \leqslant c$ or $b > c$. We proceed to prove this theorem by considering three main cases: Case 1 if $b = b'$; Case 2 if $b \leqslant c$; and Case 3 if $b > c$.

Case 1: $b = b'$.

In this case, we have $Q_3(G) = Q_3(H)$, where

$$Q_3(G) = x^{r+1} + (x+1)(x^c + x^d) - x^{r+a+b} - x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_3(H) = x^{r'+1} + (x+1)(x^{c'} + x^{d'}) - x^{r'+a+b} - x^{r'+c'+d'} - x^{a+c'} - x^{a+d'} - x^{b+c'} - x^{b+d'},$$

$c + d + r = c' + d' + r'$;

$2 \leqslant a \leqslant r + 1,\ a \leqslant b,\ a \leqslant c \leqslant d,\ a \leqslant c' \leqslant d'$, and $r < r'$.

Note that $-x^{r+c+d}$ is a term of $Q_3(G)$ and cancels with the term $-x^{r'+c'+d'}$ of $Q_3(H)$. Also $x^{\min\{r+1,c,d\}}$ and $x^{\min\{r'+1,c',d'\}}$ cannot be cancelled in $Q_3(G)$ and $Q_3(H)$, respectively. Therefore, we must have $\min\{r+1,c,d\} = \min\{r'+1,c',d'\}$. We consider two subcases: $r + 1 \leqslant c$ and $r + 1 > c$.

Subcase 1.1: $r + 1 \leqslant c$.

In this subcase, we have $\min\{r + 1, c, d\} = r + 1$ because $c \leqslant d$. Since $c' \leqslant d'$ and $r < r'$, we must have $r + 1 = c'$. Moreover $Q_4(G) = Q_4(H)$, where

$$Q_4(G) = (x + 1)(x^c + x^d) - x^{r+a+b} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_4(H) = x^{r'+1} + x^{r+2} + (x + 1)x^{d'} - x^{r'+a+b} - x^{a+r+1} - x^{a+d'} - x^{b+r+1} - x^{b+d'},$$

$$c + d = d' + r' + 1,$$

$$2 \leqslant a \leqslant r + 1,\ a \leqslant b,\ r + 1 \leqslant c \leqslant d,\ r + 1 \leqslant d',\ \text{and } r < r'.$$

The term x^{r+2} cannot be cancelled in $Q_4(H)$. Therefore, x^{r+2} is a term of $Q_4(G)$ and hence, we must have $c = r + 1$ or $c = r + 2$ or $d = r + 1$ or $d = r + 2$. Since $r + 1 \leqslant c \leqslant d$, we only need to consider the first two possibilities.

Subcase 1.1.1: $c = r + 1$.

In this subcase, we have $Q_5(G) = Q_5(H)$, where

$$Q_5(G) = x^{r+1} + (x + 1)x^d - x^{r+a+b} - x^{a+r+1} - x^{a+d} - x^{b+r+1} - x^{b+d},$$

$$Q_5(H) = x^{r'+1} + (x + 1)x^{d'} - x^{r'+a+b} - x^{a+r+1} - x^{a+d'} - x^{b+r+1} - x^{b+d'},$$

$$r + d = r' + d'; 2 \leqslant a \leqslant r + 1,\ a \leqslant b,\ r + 1 \leqslant d,\ r + 1 \leqslant d',\ \text{and } r < r'.$$

The term x^{r+1} cannot be cancelled in $Q_5(G)$. So it must also be in $Q_5(H)$. Since $r < r'$, we have $d' = r + 1$. From $r + d = r' + d'$, we get $d = r' + 1$. Moreover $Q_6(G) = Q_6(H)$, where

$$Q_6(G) = x^{d+1} - x^{r+a+b} - x^{a+d} - x^{b+d},$$

$$Q_6(H) = x^{r+2} - x^{a+b+d-1} - x^{a+r+1} - x^{b+r+1},$$

$$2 \leqslant a \leqslant r + 1,\ a \leqslant b,\ \text{and } r + 1 \leqslant d.$$

The term x^{r+2} cannot be cancelled in $Q_6(H)$. Hence, it must also be in $Q_6(G)$ which gives us $d = r + 1$. Since $d = r' + 1$, we have $r = r'$ and this contradicts our assumption.

Subcase 1.1.2: $c = r + 2$.

In this subcase, from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_7(G) = Q_7(H)$, where

$$Q_7(G) = x^{r+3} + (x + 1)x^d - x^{r+a+b} - x^{a+r+2} - x^{a+d} - x^{b+r+2} - x^{b+d},$$

$$Q_7(H) = x^{r'+1} + (x + 1)x^{d'} - x^{r'+a+b} - x^{a+r+1} - x^{a+d'} - x^{b+r+1} - x^{b+d'},$$

$$r + d + 1 = r' + d',$$

$$2 \leqslant a \leqslant r + 1,\ a \leqslant b,\ r + 2 \leqslant d,\ r + 1 \leqslant d',\ \text{and } r < r'.$$

Since the term x^{r+3} cannot be cancelled in $Q_7(G)$, we must have x^{r+3} is a term of $Q_7(H)$. Therefore, we have $r'+1=r+3$ (that is, $r'=r+2$) or $d'=r+3$ or $d'=r+2$.

Subcase 1.1.2.1: $r'=r+2$.

In this subcase, from $r+d+1=r'+d'$, we have $d=d'+1$. Moreover $Q_8(G)=Q_8(H)$, where

$$Q_8(G)=(x+1)x^d - x^{r+a+b} - x^{a+r+2} - x^{a+d} - x^{b+r+2} - x^{b+d},$$

$$Q_8(H)=(x+1)x^{d-1} - x^{r+a+b+2} - x^{a+r+1} - x^{a+d-1} - x^{b+r+1} - x^{b+d-1},$$

$$2 \leqslant a \leqslant r+1, \ a \leqslant b, \text{ and } r+2 \leqslant d.$$

The term $-x^{b+d}$ cannot be cancelled in $Q_8(G)$. Thus, we must have $-x^{b+d}$ is a term of $Q_8(H)$. Since $a \leqslant b$, $r+2 \leqslant d$, we must have $b+d=r+a+b+2$ (that is, $d=r+a+2$) and we get $Q_9(G)=Q_9(H)$, where

$$Q_9(G)=x^{r+a+3} - x^{r+a+b} - x^{a+r+2} - x^{2a+r+2} - x^{b+r+2},$$

$$Q_9(H)=-x^{2a+r+1} - x^{b+r+1} - x^{b+a+r+1}.$$

In order to have $Q_9(G)=Q_9(H)$, we must have $-x^{a+b+r+1}$ is a term of $Q_9(G)$, and this is possible only if $a+b+r+1=2a+r+2$ (that is, $b=a+1$). Thus, we get many solutions for the equation $Q(G)=Q(H)$: $a=a$, $b=a+1$, $c=r+2$, $d=a+r+2$, $r \geqslant 2$; $a'=a$, $b'=b=a+1$, $c'=r+1$, $d'=d-1=a+r+1$ and $r'=r+2$. In other words, we have

$$H=G^0_{r+2}(a,a+1;\ r+1,a+r+1) \sim G^0_r(a,a+1;\ r+2,a+r+2)=G$$

but $H \notin \mathscr{C}_r(a,a+1;\ r+2,a+r+2)$. Hence, we get the family (g).

Subcase 1.1.2.2: $d'=r+3$.

In this subcase, from $r+d+1=r'+d'$, we have $d=r'+2$. Moreover from $Q_7(G)=Q_7(H)$, we get $Q_{10}(G)=Q_{10}(H)$, where

$$Q_{10}(G)=(x+1)x^d - x^{r+a+b} - x^{a+r+2} - x^{a+d} - x^{b+r+2} - x^{b+d},$$

$$Q_{10}(H)=x^{d-1} + x^{r+4} - x^{a+b+d-2} - x^{a+r+1} - x^{a+r+3} - x^{b+r+1} - x^{b+r+3},$$

$$2 \leqslant a \leqslant r+1, \ a \leqslant b, \text{ and } r+2 < d=r'+2.$$

The term $-x^{a+r+1}$ is in $Q_{10}(H)$, but $-x^{a+r+1}$ is not a term in $Q_{10}(G)$. Thus $-x^{a+r+1}$ must be cancelled by a positive term in $Q_{10}(H)$. So we have $a+r+1=d-1$ (that is, $d=a+r+2$), or $a+r+1=r+4$ (that is, $a=3$).

If the former holds (that is, $d=a+r+2$), then we have $Q_{11}(G)=Q_{11}(H)$, where

$$Q_{11}(G)=x^{a+r+3} - x^{r+a+b} - x^{2a+r+2} - x^{b+r+2} - x^{a+b+r+2},$$

$$Q_{11}(H)=x^{r+4} - x^{2a+b+r} - x^{a+r+3} - x^{b+r+1} - x^{b+r+3},$$

$$2 \leqslant a \leqslant r+1, \text{ and } a \leqslant b.$$

The term x^{r+4} must be cancelled in $Q_{11}(H)$. This is possible only if $b=3$. Since $a \leqslant b$, we have $a=2$ or $a=3$. If $a=3$, then the term $-x^{b+r+2}=-x^{r+5}$ is in $Q_{11}(G)$, but it is not in $Q_{11}(H)$. Also, this term cannot be cancelled by a positive term in $Q_{11}(G)$. So the equation $Q(G)=Q(H)$ has no solution. For the case of $a=2$, the equation $Q(G)=Q(H)$ has a solution: $G^0_{r+2}(2,3;r+1,r+3) \sim G^0_r(2,3;r+2,r+4)$. This solution is a special case of the solution in Subcase 1.1.2.1.

If the latter holds (that is, $a=3$), then we have $Q_{12}(G)=Q_{12}(H)$, where

$$Q_{12}(G)=(x+1)x^d - x^{r+b+3} - x^{r+5} - x^{d+3} - x^{b+r+2} - x^{b+d},$$

$$Q_{12}(H)=x^{d-1} - x^{b+d+1} - x^{r+6} - x^{b+r+1} - x^{b+r+3},$$

$$3=a \leqslant b, \text{ and } r+2 < d = r'+2.$$

The term $-x^{b+d}$ cannot be cancelled in $Q_{12}(G)$; thus, this term must be in $Q_{12}(H)$. Since $d > r+2$, $-x^{b+d}$ is a term of $Q_{12}(H)$ only if $b+d=b+r+3$ (that is, $d=r+3$) or $b+d=r+6$. Note that $b+d=r+6$ also implies that $d=r+3$ because $b \geqslant 3$ and $d > r+2$. Therefore, in each case, $x^{d-1}=x^{r+2}$ cannot be cancelled in $Q_{12}(H)$, but x^{d-1} is not a term of $Q_{12}(G)$; so the equation $Q(G)=Q(H)$ has no solution.

Subcase 1.1.2.3: $d'=r+2$.

In this subcase, from $r+d+1=r'+d'$, we have $d=r'+1$. Moreover from $Q_7(G)=Q_7(H)$, we get $Q_{13}(G)=Q_{13}(H)$, where

$$Q_{13}(G)=x^{d+1} - x^{r+a+b} - x^{a+d} - x^{b+d},$$

$$Q_{13}(H)=x^{r+2} - x^{a+b+d-1} - x^{a+r+1} - x^{b+r+1},$$

$$2 \leqslant a \leqslant r+1, \ a \leqslant b, \text{ and } r+2 \leqslant d.$$

Since $r+2 \leqslant d$, there is no solution for the equation $Q(G)=Q(H)$.

Subcase 1.2: $r+1 > c$.

In this subcase, $\min\{r+1,c,d\}=c$. Recall that $\min\{r+1,c,d\}=\min\{r'+1,c',d'\}$. Therefore, $c=r'+1$ or $c=c'$. Since $c < r+1 < r'+1$, $c=r'+1$ is not possible; thus we have $c=c'$. From $Q_3(G)=Q_3(H)$, after cancelling equal terms, we have $Q_{14}(G)=Q_{14}(H)$, where

$$Q_{14}(G)=x^{r+1}+(x+1)x^d - x^{r+a+b} - x^{a+d} - x^{b+d},$$

$$Q_{14}(H)=x^{r'+1}+(x+1)x^{d'} - x^{r'+a+b} - x^{a+d'} - x^{b+d'},$$

$$d+r=d'+r'; \ 2 \leqslant a \leqslant r+1, \ a \leqslant b, \ c < r+1, \ c \leqslant d,$$

$$c=c' \leqslant r'+1, \ c \leqslant d', \text{ and } r < r'.$$

Now $\min\{r+1,d\}=\min\{r'+1,d'\}$. If $\min\{r+1,d\}=r+1$, then $r+1=d'$ because $r<r'$. Since $d+r=d'+r'$, we have $d=r'+1$. Proceed as in Subcase 1.1.1, we will get $r=r'$, which contradicts our assumption. If $\min\{r+1,d\}=d$, then $d=r'+1$ or $d=d'$. Since $d \leqslant r+1 < r'+1$, $d=r'+1$ is impossible. Also since $d+r=d'+r'$, the case of $d=d'$ implies $r=r'$, which contradicts our assumption $r<r'$. Thus, the equation $Q(G)=Q(H)$ has no solution.

Case 2: $b \leqslant c$ $(b \neq b')$.

In this case, the equation $Q(G)=Q(H)$ has no solution.

Case 3: $b \geqslant c$ $(b \neq b')$.

In this case, the equation $Q(G)=Q(H)$ has a solution only when $r+1 \leqslant c$, $c=d'$, and $b=r+2$. The solution is $a=a$, $b=r+2$, $c=a+1$, $d=a+r+2$, $r \geqslant 2$; $a'=a$, $b'=a+r+1$, $c'=r+1$, $d'=a+1$ and $r'=r+2$. In other words, we have

$$H=G^0_{r+2}(a,a+r+1;\ r+1,a+1) \sim G^0_r(a,r+2;\ a+1,a+r+2)=G,$$

but $H \notin \mathscr{C}_r(a,r+2;\ a+1,a+r+2)$. This solution gives us the family (f).

The proof for Cases 2 and 3 above are similar to that of Case 1. The detail proof can be obtained by e-mail from the second author or view at http://www.fsas.upm.edu.my/yhpeng/publish/p3c23.pdf. □

From Theorems 1, E, and H, we have the following result.

Theorem 2. *If* $r \geqslant 2$ *and* $\min\{a,b,c,d\} \geqslant 2$, *then the family of graphs* $\mathscr{C}_r(a,b;\ c,d)$ *is a chromatic equivalence class except those graphs listed in Theorem* 1.

Theorems F and G and the corollary of Theorem H together with Theorem 1 completely determine the chromatic equivalence classes of any $G^s_t(a,b;\ c,d)$. Hence Problem 2 of [3] is solved.

Theorem 3. *The chromatic equivalence classes are all single* $\mathscr{C}_r(a,b;\ c,d)$ *with the exception of the following unions of* $\mathscr{C}_r(a,b;\ c,d)$.

(a) $\mathscr{C}_0(1,b;\ c,d) \cup \mathscr{C}_{b-1}(1,c;\ 1,d) \cup \mathscr{C}_{c-1}(1,b;\ 1,d) \cup \mathscr{C}_{d-1}(1,b;\ 1,c)$, *for* $b,c,d \geqslant 2$;
(b) $\mathscr{C}_r(1,b;\ c,d) \cup \mathscr{C}_{c-1}(1,b;\ r+1,d) \cup \mathscr{C}_{d-1}(1,b;\ c,r+1)$, *for* $r \geqslant 1$ *and* $b,c,d \geqslant 2$, *except for* $r=2$ *and* $b=d=c+1$;
(c) $\mathscr{C}_0(2,b;\ b+1,b+2) \cup \mathscr{C}_2(1,b+1;\ b,b+1) \cup \mathscr{C}_{b-1}(1,b+1;\ 3,b+1) \cup \mathscr{C}_b(1,b+1;\ 3,b)$, *for any* $b \geqslant 2$;
(d) $\mathscr{C}_1(3,5;\ 5,8) \cup \mathscr{C}_5(2,6;\ 4,5)$;
(e) $\mathscr{C}_1(3,3;\ c,c+2) \cup \mathscr{C}_{c-1}(2,4;\ 3,c+1)$, *for any* $c \geqslant 3$;
(f) $\mathscr{C}_1(3,b;\ 3,b+2) \cup \mathscr{C}_{b-1}(2,b+1;\ 3,4)$, *for any* $b \geqslant 3$;
(g) $\mathscr{C}_r(r+2,b;\ b+1,b+r+2) \cup \mathscr{C}_{r+2}(r+1,b+1;\ b,b+r+1)$, *for any* $b \geqslant r+2 \geqslant 2$ *or* $r=1$ *and* $b \geqslant 2$;
(h) $\mathscr{C}_r(r+2,c+r+2;\ c,c+1) \cup \mathscr{C}_{r+2}(r+1,c+r+1;\ c,c+1)$, *for any* $c \geqslant r+2 \geqslant 2$ *or* $r=1$ *and* $c \geqslant 2$.

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