Star Coloring and Tree-width of the Kneser Graph $KG(n,2)$

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Abstract

A proper coloring of the vertices of a graph G is called a star coloring if any path of length three in G is not 2-colored. The star chromatic number of G is the minimum number of colors required to obtain a star coloring of G. In this paper, we give the exact value of the star chromatic number of the Kneser graph $KG(n,2)$. Moreover, we obtain a lower bound and an upper bound for the tree-width of these graphs.

Keywords: Star coloring; Tree-width; Kneser graph.

1 Introduction

Throughout this paper, all graphs are finite and simple. We use $u \sim v$ and $u \nsim v$ to respectively denote the adjacency and non-adjacency relations between vertices u and v. We denote a path and a cycle on n vertices by P_n and C_n, respectively. We refer the reader to [12] for graph-theoretic notation and terminology not described in this paper.

A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. Given a proper coloring c of graph G and any subset $A \subseteq V(G)$, we let $c(A)$ denote the set of all colors used to color vertices in A. A proper vertex coloring of a graph G is called a *star coloring* if any path of length three in G is not 2-colored; equivalently, the union of every two color classes in G induces a forest whose components are stars. The *star chromatic number* of

G, denoted by $\chi_s(G)$, is the minimum number of colors required to obtain a star coloring of G. Star colorings of graphs were introduced by Grünbaum in 1973 [4] (see also [1, 3, 9]).

The *Kneser graph* $KG(n, k)$, $n \geq 2k$, is a graph whose vertices are the k-element subsets of an n-element set, where two vertices are adjacent if and only if the two corresponding sets are disjoint. In this paper, vertices of $KG(n, 2)$ are the 2-element subsets $\{i, j\}$, $1 \leq i < j \leq n$, which we denote ij.

Kneser graphs have many interesting properties and have been the subject of much research. It was conjectured by Kneser in 1955 [7] and proved by Lovász in 1978 [8] that $\chi(KG(n, k)) = n - 2k + 2$. Since then several types of colorings of Kneser graphs have been considered. For example, the circular chromatic number, the b-chromatic number and the multichromatic number of Kneser graphs were investigated in [5], [6] and [11], respectively.

The concept of tree-width of a graph introduced by Robertson and Seymour in 1984 [10] to measure how tree-like a graph behaves. Fertin et al. [3] gave the bound $\binom{t+2}{2}$ on the star chromatic number of graphs with tree-width t.

In this paper, we obtain the exact value of the star chromatic number of the Kneser graph $KG(n, 2)$, for $n \geq 5$. We also give a lower bound and an upper bound on the tree-width of $KG(n, 2)$.

2 Main results

In this section, we show that $\chi_s(KG(n, 2)) = \binom{n-1}{2}$, for $n \geq 6$ and $\chi_s(KG(5, 2)) = 5$. First, we prove the following proposition.

Proposition 1. *Let $G = (X, Y)$ be a bipartite graph, such that $|X| = |Y| = n$, and each vertex in G has degree $n - 1$. Then, $\chi_s(G) = n$. Moreover, there are only two types of optimum star colorings of G.*

Proof. Let c be a star coloring of G. If the vertices of X or Y all receive different colors, then the number of colors used in c is at least n.

Otherwise, there are vertices $u_1, u_2 \in X$ and $v_1, v_2 \in Y$, where $c(u_1) = c(u_2)$ and $c(v_1) = c(v_2)$. Since there is no 2-colored P_4 and G is $(n - 1)$-regular, neither u_1 nor u_2 is adjacent to both vertices v_1 and v_2. Say

$u_1 \nsim v_1$ and $u_2 \nsim v_2$. Therefore, there are no other pairs of vertices with the same color in the same part (except u_1, u_2 and v_1, v_2), because all vertices in Y (similarly in X), except v_1 (u_1), are adjacent to u_1 (v_1), and (G, c) does not have a 2-colored P_4. Hence, in such a star coloring of G, at least $2 + (n-2) = n$ colors are needed. Thus, $\chi_s(G) \geq n$. In Figure 1, two possible types of optimum star colorings of G with n colors are shown. Hence, $\chi_s(G) = n$. ■

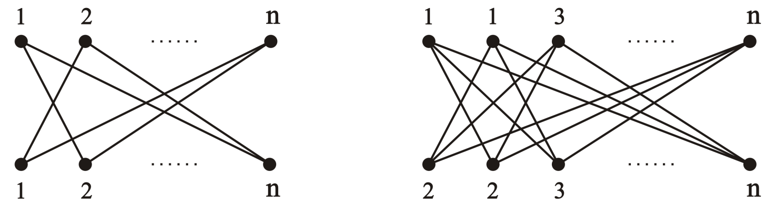

Figure 1: Two optimum star colorings of G.

Note that, for every $n \geq 6$, the vertex set of the Kneser graph $KG(n, 2)$ can be decomposed into three subsets $\{12\}$, A and B; where A and B are the sets of all vertices adjacent and non-adjacent to vertex 12, respectively. It is easy to see that, $|A| = \binom{n-2}{2}$, and the induced subgraph on B is a bipartite graph with parts $X := \{1i : 3 \leq i \leq n\}$ and $Y := \{2i : 3 \leq i \leq n\}$.

By Proposition 1, $\chi_s(B) = n-2$. Thus, we can give a color assignment to the vertices of B, the same as the first star coloring given in the proposition above (Figure 1). Then, we assign $\binom{n-2}{2}$ new colors to the vertices of A, and, finally, give one of the colors of the vertices of B, to the vertex 12. It can be easily checked that, this coloring is a star coloring of $KG(n, 2)$ with $\binom{n-2}{2} + n - 2 = \binom{n-1}{2}$ colors. Thus, $\chi_s(KG(n, 2)) \leq \binom{n-1}{2}$. Further, since $KG(n, 2)$ has $\binom{n}{2}$ vertices, in a given optimum star coloring c of $KG(n, 2)$, there are at least two vertices with a same color. Without loss of generality (after a renaming if necessary), suppose that $c(13) = c(23) = a$.

To prove the main result of this section, we need the following lemmas, which will make use of the above notation. Specifically, A, B, X, Y, and c continue to be used throughout this section.

Lemma 1. *Let $A' := \{kl : 4 \leq k < l \leq n\}$ together with $A'' := \{3l : 4 \leq l \leq n\}$ be a partition of A. In the optimum star coloring c of graph $KG(n, 2)$ we have*

(I) $c(B) \cap c(A') = \emptyset$.

(II) $|c(A')| = \binom{n-3}{2}$.

Proof. (I) The color of any vertex in $X \cup Y$ can not be the same as any vertex in A', otherwise there would be a 2-colored P_4 containing 13 and 23, which is a contradiction.

(II) Since all vertices in A' are adjacent to 13 and 23, there are no two vertices in A' with the same color, otherwise there would be a 2-colored P_4 containing 13 and 23, which contradicts that c is a star coloring. ■

Let t be the number of pairs $(1i, 2i)$, $i \geq 4$, of vertices in B, such that $c(1i) = c(2i)$. Then we have the following lemma.

Lemma 2. *If $t \geq 1$, or equivalently there is some i, $4 \leq i \leq n$, such that $c(1i) = c(2i)$, then we have the following facts.*

(I) *All vertices in $A'' \backslash \{3i\}$ must have different colors. In other words, only for one $l \geq 4$, $l \neq i$, it is possible that $c(3l) = c(3i)$, i.e., a repeated color in A''.*

(II) *If $c(A') \cap c(A'') \neq \emptyset$, then $c(3l) = c(il)$ or $c(3i) = c(li)$, for some $l \geq 4$.*

(III) *If $c(A'') \cap c(B) \neq \emptyset$, then either $c(3i) = c(1i) = c(2i)$ or $c(3i) = c(13) = c(23) = a$.*

(IV) *If $c(A') \cap c(A'') \neq \emptyset$ and there exists $l \geq 4$, such that $c(3l) = c(il)$, then $c(12) \notin \{a, c(1i)\}$.*

Proof. (I) Otherwise, there are $k, k' \neq i$ (noting that $n \geq 6$), such that $c(3k) = c(3k')$ and the path $3k$ $1i$ $3k'$ $2i$ is 2-colored.

(II) Otherwise, if there exists k and l, with $l, k \neq i$ and $c(3l) = c(kl)$, then the path $3l$ $1i$ kl $2i$ is 2-colored.

(III) Otherwise, suppose that there is $l \neq i$, and $c(3l) = c(1l)$ (similarly $c(3l) = c(2l)$). Then $1i$ $3l$ $2i$ $1l$ is 2-colored ($2i$ $3l$ $1i$ $2l$ is 2-colored). If there exists $l \neq i$, and $c(3l) = c(13) = c(23) = a$, then the path 13 $2i$ $3l$ $1i$ is 2-colored.

(IV) If there exists $l \geq 4$, such that $c(3l) = c(il)$ and $c(12) \in \{a, c(1i)\}$, then the path 13 il 12 $3l$ or the path $1i$ $3l$ 12 il is 2-colored. ■

Now we are ready to prove our main theorem, the proof of which uses the above notation.

Theorem 1. *For $n \geq 6$, $\chi_s(KG(n,2)) = \binom{n-1}{2}$, and $\chi_s(KG(5,2)) = 5$.*

Proof. Let $n \geq 6$. We have already shown that $\chi_s(KG(n,2)) \leq \binom{n-1}{2}$. To prove the equality we need to show that every (optimum) star coloring of $KG(n,2)$ requires at least $\binom{n-1}{2}$ colors. We consider the following two possibilities.

Case 1. In the optimum star coloring c, there are no two vertices with a same color in the same part of B.

If $t = 0$, then by Lemma 1 (I) and (II), we conclude that

$$\begin{aligned} |c(KG(n,2))| &\geq |c(B)| + |c(A')| \\ &= (2(n-3)+1) + \binom{n-3}{2} \\ &= \binom{n-1}{2}. \end{aligned}$$

If $t = 1$, and $c(1i) = c(2i) = b$, $4 \leq i \leq n$, then there are two possibilities.

If $c(A'') \cap c(B) \neq \emptyset$. Then, by Lemma 2 (III), we have either $c(3i) = c(1i) = c(2i) = b$ or $c(3i) = c(13) = c(23) = a$. Now, if for some $k \geq 4$, $k \neq i$, $c(12) = c(1k)$ (similarly, $c(12) = c(2k)$), then, in the former case, the path $2i$ $1k$ $3i$ 12 (similarly, $1i$ $2k$ $3i$ 12) is 2-colored, and, in the latter case, the path 23 $1k$ $3i$ 12 (similarly, 13 $2k$ $3i$ 12) is 2-colored. Therefore, $c(12) \cap c(B) \subset \{a, b\}$. On the other hand, by Lemma 2 (II) and (IV), if $c(12) \in \{a, b\}$ then $c(A') \cap (c(A'') \backslash c(3i)) = \emptyset$. Thus, we have

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A')| + |c(B)| + |c(A'' \backslash \{3i\}) \backslash c(B)| \\ &\geq \binom{n-3}{2} + (2(n-4)+2) + (n-4) \\ &= \binom{n-1}{2} + n - 5 > \binom{n-1}{2} \quad (n \geq 6). \end{aligned}$$

This contradicts the assumption that c is an optimum star coloring. Hence, $c(12) \cap c(B) = \emptyset$ and consequently $c(12) \cap c(B \cup A) = \emptyset$, so we have

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A')| + |c(B)| + |c(12)| \\ &= \binom{n-3}{2} + (2(n-4)+2) + 1 \\ &= \binom{n-1}{2}. \end{aligned}$$

If $c(A'') \cap c(B) = \emptyset$, then either there is a vertex in A'' which has a different color to the vertices in A', and we have

$$
\begin{aligned}
|c(KG(n,2))| &\geq |c(A')| + |c(B)| + |c(A'') \backslash c(A')| \\
&\geq \binom{n-3}{2} + (2(n-4)+2) + 1 \\
&= \binom{n-1}{2}
\end{aligned}
$$

or, for each $l \geq 4$, $c(3l) = c(il)$ (by Lemma 2 (II)). We will now show that $c(12) \cap c(B) = \emptyset$. By Lemma 2 (IV), $c(12) \notin \{a, b\}$. If for some $k \geq 4$, $k \neq i$, $c(12) = c(1k)$ (similarly $c(12) = c(2k)$), then since $n \geq 6$, there exists $k' \geq 4$, $k' \notin \{i, k\}$, such that the path $3k'$ 12 ik' $1k$ is 2-colored. Thus, we have

$$
\begin{aligned}
|c(KG(n,2))| &\geq |c(A')| + |c(B)| + |c(12)| \\
&\geq \binom{n-3}{2} + (2(n-4)+2) + 1 \\
&= \binom{n-1}{2}.
\end{aligned}
$$

If $t = 2$, and there exist $i, j \geq 4$, $i \neq j$, such that $c(1i) = c(2i) = b$ and $c(1j) = c(2j) = c$, then by Lemma 2 (I) and (II), only the vertices $3i$, $3j$, and ij may receive the same color in A. Also, by Lemma 2 (III), we can easily deduce that $c(A'') \cap c(B) = \emptyset$. Now, there are the following possibilities.

If $c(ij) = c(3i) = c(3j)$, then $c(12) \cap c(B) = \emptyset$. Since, by Lemma 2 (IV), $c(12) \notin \{a, b, c\}$. Also, if for some $k \geq 4$, $k \neq i, j$, $c(12) = c(1k)$ (similarly, $c(12) = c(2k)$), then the path $1k$ $3i$ 12 ij is 2-colored (similarly, $2k$ $3i$ 12 ij is 2-colored). Therefore, the color of 12 is distinct from $c(A)$ and $c(B)$, and we have

$$
\begin{aligned}
|c(KG(n,2))| &\geq |c(A)| + |c(B)| + |c(12)| \\
&\geq \left(\binom{n-2}{2} - 2\right) + (2(n-5)+3) + 1 \\
&= \binom{n-1}{2} + n - 6 \geq \binom{n-1}{2} \quad (n \geq 6).
\end{aligned}
$$

If $c(ij)$, $c(3i)$, and $c(3j)$ are not pairwise equal, then

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A)| + |c(B)| \\ &\geq \left(\binom{n-2}{2} - 1\right) + (2(n-5)+3) \\ &= \binom{n-1}{2} + n - 6 \geq \binom{n-1}{2} \quad (n \geq 6). \end{aligned}$$

If $t \geq 3$, then by Lemma 2 (I), (II), and (III), we deduce that all vertices in A must have different colors and $c(A) \cap c(B) = \emptyset$. Thus

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A)| + |c(B)| \\ &\geq \binom{n-2}{2} + n - 2 \\ &= \binom{n-1}{2}. \end{aligned}$$

Case 2. In the optimum star coloring c of $KG(n,2)$, there are two vertices in the same part of B with the same color. Without loss of generality, assume that $c(14) = c(15) = b$.

Now, we consider a new decomposition $(A_1', A_1'', B', \{45\})$ for $KG(n,2)$, which is isomorphic to $(A', A'', B, \{12\})$, where $A_1' := \{kl : 6 \leq k < l \leq n\} \cup (Y \setminus \{24, 25\}) \cup \{3k : k \geq 6\}$ (corresponding to A'), $A_1'' := (X \setminus \{14, 15\}) \cup \{12\}$ (corresponding to A''), and $B' := \{4k : 1 \leq k \leq n, k \neq 4\} \cup \{5k : 1 \leq k \leq n, k \neq 5\}$ (corresponding to B). In other words, $A_1' \cup A_1''$ and B' are the sets of all vertices adjacent and non-adjacent to vertex 45, respectively. Further, the induced subgraph on B' is a bipartite graph with parts $X' := \{4i : 1 \leq i \leq n, i \neq 4\}$ and $Y' := \{5i : 1 \leq i \leq n, i \neq 5\}$.

We need the following lemma.

Lemma 3. *If in the star coloring c of $KG(n,2)$, $c(14) = c(15) = b$, then we have*

(I) $c(36) \notin c(A_1' \cup B' \cup \{45, 12\})$.

(II) *If $c(34)$ and $c(35)$ are not repeated in X' and Y' respectively, then* $|c(KG(n,2))| \geq \binom{n-1}{2}$.

(III) *If $c(34)$ and $c(35)$ are repeated in X' and Y' respectively, then $c(45)$ is not repeated in $KG(n,2)$.*

(IV) $|c(B')| = 2(n-5) + |\{b, c(24), c(25)\}| + |c(\{c(34), c(35)\}) \setminus c(B' - \{c(34), c(35)\})|$.

Proof. (I) Otherwise, if for some $k \geq 6$, $c(36) = c(2k)$, then the path $2k$ 14 36 15 is 2-colored. If $c(36) = a$, then the path 23 14 36 15 is 2-colored. If for some $k, l \geq 6$, $c(36) = c(kl)$, then the path kl 14 36 15 is 2-colored. If for some $k > 6$, $c(36) = c(3k)$, then the path $3k$ 14 36 15 is 2-colored. By definition of a Kneser graph, $c(36) \notin \{b, c(24), c(25)\}$. If $c(36) = c(34)$ or $c(36) = c(35)$, then the path 34 15 36 14 or 35 14 36 15, respectively, is a 2-colored P_4. If for some $k \geq 6$, $c(36) = c(4k)$ or $c(36) = c(5k)$, then the path $4k$ 15 36 14 or $5k$ 14 36 15, respectively, is a 2-colored P_4. Finally, 36 is adjacent to both 12 and 45. Therefore, $c(36) \notin c(A'_1 \cup B' \cup \{45, 12\})$.

(II) By assumption and by Lemma 1 (I) and (II), for such a decomposition, all the vertices in $B' \cap A'$ ($\subset A'$) have different colors which are distinct from $\{b, c(24), c(25), c(34), c(35)\}$. This implies that there are no two vertices in the same part of B' which have the same color. Therefore, by renaming the vertices, it is an instance of Case 1 which already proved.

(III) Note that $45 \in A'$ and by Lemma 1 (II), all the vertices in A' have different colors. Therefore, $c(45) \cap c(B' \backslash B) \subset \{c(34), c(35)\}$. Now, either $c(4k) = c(34)$ for some $k \geq 6$, or $c(24) = c(34)$. Similarly, either $c(5l) = c(35)$ for some $l \geq 6$, or $c(25) = c(35)$. But, $4k, 5l, 45 \in A'$ and $24, 25 \in B$. Thus, by Lemma 1 (I) and (II), in all cases, $c(45) \cap c(B' \backslash B) = \emptyset$. Also, 45 is adjacent to 12 and to all the vertices in $A \backslash (B' \backslash B)$. Moreover, since $45 \in A'$, by Lemma 1 (I), $c(45) \cap c(B) = \emptyset$. Hence, $c(45)$ is not repeated in $KG(n, 2)$.

(IV) By Lemma 1 (I) and (II), all the vertices in $B' \backslash (B \cup \{34, 35\})$ ($\subset A'$) have different colors and $\{b, c(24), c(25)\} \cap c(B' \backslash (B \cup \{34, 35\})) = \emptyset$. Also, $c(\{34, 35\}) \cap c(B' \backslash \{34, 35\}) \neq \emptyset$ occurs only when for some $k, l \geq 6$, $c(4k) = c(34)$, or $c(5l) = c(35)$, or $c(24) = c(34)$, or $c(25) = c(35)$. ∎

Now, we have two possibilities $c(34) = c(35)$ or $c(34) \neq c(35)$.

If $c(34) = c(35) = d$, then d can not be repeated in B', and therefore, by Lemma 3 (II), we have $|c(KG(n, 2))| \geq \binom{n-1}{2}$.

If $c(34) = e \neq f = c(35)$ and both e and f are not repeated in B', then by Lemma 3 (II), $|c(KG(n, 2))| \geq \binom{n-1}{2}$. If at least one of e or f is repeated in B', then we have two possibilities; either $c(24) = c(25)$ or $c(24) \neq c(25)$.

If $c(24) = c(25) = c$, then we can have $c(\{34, 35\}) \cap c(B' \backslash \{34, 35\}) \neq \emptyset$ only possibly when for some $l, l' \geq 6$, $c(4l) = c(34)$ or $c(5l') = c(35)$. Now, we can see that $c(12) \notin c(A'_1) \cup c(B') \cup \{c(45)\}$. Otherwise, if $c(12) = a$, then 13 $4l$ 12 34 or 13 $5l'$ 12 35 is a 2-colored P_4. If $c(12) = b$, then $4l$ 12 34 15 or $5l'$ 12 35 14 is a 2-colored P_4 (similarly, $c(12) \neq c$). If $c(12) = c(2k)$, for some $k \geq 6$, then $4l$ 12 34 $2k$ or $5l'$ 12 35 $2k$ is a 2-colored P_4.

Thus, by Lemma 1 (II), and using Lemma 3 (I), (III), and (IV), we conclude that

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A'_1)| + |c(B') \cup \{c(45)\}| + |\{c(36), c(12)\}| \\ &\geq \binom{n-3}{2} + (2(n-5)+3) + 2 \\ &= \binom{n-1}{2} \quad (n \geq 6). \end{aligned}$$

If $c(24) = c \neq d = c(25)$, then by Lemma 1 (II) and Lemma 3 (I), (III), and (IV), we have

$$\begin{aligned} |c(KG(n,2))| &\geq |c(A'_1)| + |c(B') \cup \{c(45)\}| + |c(36)| \\ &\geq \binom{n-3}{2} + (2(n-5)+4) + 1 \\ &= \binom{n-1}{2} \quad (n \geq 6). \end{aligned}$$

Finally, we consider $KG(5,2)$, which is the well-known Petersen graph. Note that, Petersen graph contains C_5. According to a proposition by Fertin et al. ([3], Proposition 3.2), $\chi_s(C_5) = 4$. Thus, $\chi_s(KG(5,2)) \geq 4$. By inspection, we find that $\chi_s(KG(5,2)) \neq 4$. In Figure 2, we present a star coloring of $KG(5,2)$ with 5 colors. Hence, $\chi_s(KG(5,2)) = 5$. ■

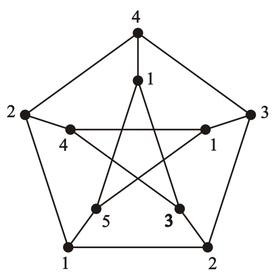

Figure 2: An optimum star coloring of Petersen graph.

3 Tree-width of $KG(n,2)$

In this section, we obtain a lower bound and an upper bound for the tree-width of $KG(n,2)$.

We denote the tree-width of a graph G with $\mathrm{tw}(G)$. A *chordal graph* is a graph without induced cycles of order more than 3. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of maximum clique of G. It is well known (see [2], Corollary 12.3.9) that the tree-width of a graph G can be expressed as

$$\mathrm{tw}(G) = \min\{\omega(H) - 1 : E(G) \subseteq E(H) \text{ and } H \text{ is chordal}\}. \tag{1}$$

We also know the following theorem by Albertson et al. [1] and Fertin et al. [3].

Theorem 2. *If a graph G is of tree-width at most t, then $\chi_s(G) \leq \binom{t+2}{2}$.*

The following corollary gives a lower bound for the tree-width of $KG(n, 2)$.

Corollary 1. *For each integer $n \geq 6$, we have $\mathrm{tw}(KG(n, 2)) \geq n - 3$.*

Proof. Let $\mathrm{tw}(KG(n, 2)) = t$, for $n \geq 6$. By Theorems 1 and 2, we have $\chi_s(KG(n, 2)) = \binom{n-1}{2} \leq \binom{t+2}{2}$. Thus $t + 2 \geq n - 1$, and consequently $t \geq n - 3$. ■

Using the decomposition of the vertices of $KG(n, 2)$, as mentioned in the proof of Theorem 1, we find an upper bound for the tree-width of $KG(n, 2)$, as follows.

Theorem 3. *For each integer $n \geq 5$, we have $\mathrm{tw}(KG(n, 2)) \leq \binom{n-1}{2} - 1$.*

Proof. By (1), it suffices to find a chordal graph with clique number $\binom{n-1}{2}$, which contains $KG(n, 2)$. We obtain such a chordal graph H, by the given vertex partition in the proof of Theorem 1 as follows. Suppose that A and B are the sets of all vertices adjacent and non-adjacent to vertex 12, respectively. Thus, $B = (X, Y)$ with $X = \{1j : 3 \leq j \leq n\}$ and $Y = \{2j : 3 \leq j \leq n\}$.

Let $V(H) = V(KG(n, 2))$ and $E(H) = E(KG(n, 2)) \cup \{uv : u, v \in A \cup X\}$. Now, it can be easily checked that

$$\begin{aligned} \omega(H) = |A| + |X| &= \binom{n-2}{2} + (n-2) \\ &= \binom{n-1}{2}. \end{aligned}$$

Moreover, every cycle of order more than 3 in H contains at least two non-successive vertices in $A \cup X$, and hence is not an induced cycle. Thus, H is a chordal graph. ■

We use the software for computing the tree-width of graphs at `http://treewidth.com` to observe that the equality holds in Theorem 3 for $6 \leq n \leq 14$.

n	$\binom{n-1}{2} - 1$	$\mathrm{tw}(KG(n,2))$
5	5	4
6	9	9
7	14	14
8	20	20
9	27	27
10	35	35
11	44	44
12	54	54
13	65	65
14	77	77
15	90	89

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