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EDGE CLIQUE COVERING SUM OF GRAPHS

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Abstract. The edge clique cover sum number (resp. edge clique partition sum number) of a graph G, denoted by $\mathrm{scc}(G)$ (resp. $\mathrm{scp}(G)$), is defined as the smallest integer k for which there exists a collection of complete subgraphs of G, covering (resp. partitioning) all edges of G such that the sum of sizes of the cliques is at most k. By definition, $\mathrm{scc}(G) \leqq \mathrm{scp}(G)$. Also, it is known that for every graph G on n vertices, $\mathrm{scp}(G) \leqq n^2/2$. In this paper, among some other results, we improve this bound for $\mathrm{scc}(G)$. In particular, we prove that if G is a graph on n vertices with no isolated vertex and the maximum degree of the complement of G is $d-1$, for some integer d, then $\mathrm{scc}(G) \leqq cnd \lceil \log((n-1)/(d-1)) \rceil$, where c is a constant. Moreover, we conjecture that this bound is best possible up to a constant factor. Using a well-known result by Bollobás on set systems, we prove that this conjecture is true at least for $d = 2$. Finally, we give an interpretation of this conjecture as an interesting set system problem which can be viewed as a multipartite generalization of Bollobás' two families theorem.

1. Introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph G, we mean a subset of mutually adjacent vertices of G as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices. Also, a *biclique* of G is a complete bipartite subgraph of G. A *clique* (resp. *biclique*) *covering* of G is defined as a family of cliques (resp. bicliques) of G such that every edge of G lies in at least one of the cliques (resp. bicliques) comprising this family. A clique (resp. biclique) covering in which each edge belongs to exactly one clique (resp. biclique), is called a *clique* (resp. *biclique*) *partition.* The minimum size of a clique covering, a biclique covering, a clique partition and a biclique partition of G are called

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clique cover number, biclique cover number, clique partition number and *biclique partition number* of G and are denoted by $\mathrm{cc}(G)$, $\mathrm{bc}(G)$, $\mathrm{cp}(G)$ and $\mathrm{bp}(G)$, respectively.

The subject of clique covering has been widely studied in recent decades. First time, Erdős et al. in [6] presented a close relationship between the clique covering and the set intersection representation. Also, they proved that the clique partition number of a graph on n vertices cannot exceed $n^2/4$ (known as Erdős–Goodman–Pósa theorem). The connections of clique covering and other combinatorial objects have been explored (see e.g. [17, 21]). For a survey of the classical results on the clique and biclique coverings see [13,16].

A number of different variants of the clique and biclique covering number have been investigated in the literature. For instance, the *local clique* (resp. *biclique*) *cover number* denoted by $\mathrm{lcc}(G)$ (resp. $\mathrm{lbc}(G)$) is the least number r such that G admits a clique (resp. biclique) covering where no vertex is in more than r of these cliques (resp. bicliques). Erdős and Pyber [7] proved that $\mathrm{lbc}(G) \leqq cn/\log n$, for some constant c. For more recent results on $\mathrm{lcc}(G)$ and $\mathrm{lbc}(G)$ see [11] and [15], respectively.

Moreover, Chung et al. in [4] and independently Tuza in [20] considered a weighted version of the biclique covering. In fact, given a graph G, they were concerned with minimizing $\sum_{B\in\mathcal{B}} \left|V(B)\right|$ among all biclique coverings $\mathcal{B}$ of G. They proved that every graph on n vertices has a biclique covering such that the sum of the number of vertices of these bicliques is $O(n^2/\log n)$ [4,20]. Furthermore, a clique counterpart of weighted biclique cover number has been studied. It was conjectured by Katona and Tarján and was proved independently by Chung [3], Győri and Kostochka [8] and Kahn [12] that every graph on n vertices admits a clique partition such that the sum of the number of vertices in these cliques is at most $n^2/2$. This can be considered as a generalization of the Erdős–Goodman–Pósa theorem.

In this paper, we are concerned with the weighted version of the clique cover number. Let G be a graph. The *edge clique cover sum number* of G, denoted by $\mathrm{scc}(G)$, is defined as the minimal integer k for which there exists a clique covering $\mathcal{C}$ of G, such that the sum of its clique sizes is at most k. For a clique covering $\mathcal{C}$ of a graph G and a vertex $u \in V(G)$, let the *valency* of u (with respect to $\mathcal{C}$), denoted by $\mathcal{V}_{\mathcal{C}}(u)$, be the number of cliques in $\mathcal{C}$ containing u. In fact,

$$\mathrm{scc}(G) = \min_{\mathcal{C}} \sum_{C\in\mathcal{C}} |C| = \min_{\mathcal{C}} \sum_{u\in V(G)} \mathcal{V}_{\mathcal{C}}(u),$$

where the minimum is taken over all clique coverings of G. Analogously, one can define the *edge clique partition sum number* of G, denoted by $\mathrm{scp}(G)$. As a matter of fact, the above-mentioned result in [3,8,12] states that for every graph G on n vertices, $\mathrm{scc}(G) \leqq \mathrm{scp}(G) \leqq n^2/2$.

In order to reveal inherent difference between $\mathrm{cc}(G)$ and $\mathrm{scc}(G)$, we introduce a similar parameter $\mathrm{scc}'(G)$ which is defined as the minimum of the sum of clique sizes in a clique covering $\mathcal{C}$ achieving $\mathrm{cc}(G)$, i.e.

$$\mathrm{scc}'(G) := \min\left\{ \sum_{C\in\mathcal{C}} |C| : \mathcal{C} \text{ is a clique covering of } G \text{ and } |\mathcal{C}| = \mathrm{cc}(G)\right\}.$$

It is evident that $\mathrm{scc}(G) \leqq \mathrm{scc}'(G)$. In Section 2, first in Theorem 1, we will see that for some classes of graphs G, the quotient $\mathrm{scc}'(G)/\mathrm{scc}(G)$ can be arbitrary large. Then, we give some general bounds on the edge clique cover sum number and the edge clique partition sum number. In particular, we prove that if G is a graph on n vertices with no isolated vertex and the maximum degree of the complement of G is $d-1$, for some integer d, then $\mathrm{scc}(G) \leqq cnd\left\lceil \log\big((n-1)/(d-1)\big)\right\rceil$, where c is a constant. We conjecture that this upper bound is best possible up to a constant factor. In Section 3, using a well-known result by Bollobás, we prove the correctness of this conjecture for $d=2$. Moreover, we show that for every even integer n, if G is the complement of an induced matching on n vertices, then $\mathrm{scc}(G) \sim n\log n$, where $f \sim g$ means that f asymptotically approaches to g. Finally, in Section 4, we give an interpretation of this conjecture as an interesting set system problem which can be viewed as a multipartite generalization of Bollobás' two families theorem.

2. Some bounds

In this section, first we present a class of graphs for which the family of clique coverings achieving $\mathrm{cc}(G)$ is disjoint from the family of clique coverings achieving $\mathrm{scc}(G)$. Then, we provide several inequalities relating the introduced clique covering parameters. Moreover, we present an upper bound for $\mathrm{scc}(G)$ in terms of the number of vertices and the maximum degree of the complement of G.

THEOREM 1. *There exists a sequence of graphs $\{G_n\}$ such that $\mathrm{scc}'(G_n)/\mathrm{scc}(G_n)$ tends to infinity as n tends to infinity.*

PROOF. Let n be a positive integer and G_n be a graph on $3n+2$ vertices, such that $V(G_n) = \{x_0, y_0\} \cup X \cup Y \cup Z$, where $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ and $Z = \{z_1, \dots, z_n\}$ and adjacency is as follows. The sets $X \cup \{x_0\}$, $Y \cup \{y_0\}$ and Z are three cliques and every vertex in Z is adjacent to every vertex in $X \cup Y$. Moreover, for all $i, j \in \{1, \dots, n\}$, x_i is adjacent to y_j if and only if $i = j$ (see Figure 1 for a schematic form of G_n).

First, note that each clique of G_n covers at most one edge from the set $\{x_iy_i : 1 \leqq i \leqq n\} \cup \{x_0x_1, y_0y_1\}$. This yields $\mathrm{cc}(G_n) \geqq n+2$. Now, we show that G_n has a unique clique covering containing exactly $n+2$ cliques. Let

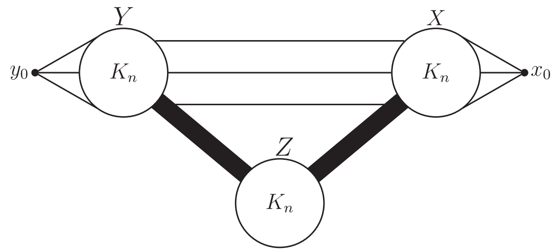

Fig. 1: A schematic form of the graph G_n

$\mathcal{C}$ be a clique covering of G_n consisting of $n+2$ cliques. Assume that the clique $C_i \in \mathcal{C}$ covers the edge x_iy_i, for $1 \leqq i \leqq n$, and the cliques $C_{n+1} \in \mathcal{C}$ and $C_{n+2} \in \mathcal{C}$ cover the edges y_0y_1 and x_0x_1, respectively. Note that $C_{n+2} \subseteqq \{x_0\} \cup X$ and $x_0 \notin \cup_{i=1}^{n+1} C_i$. Therefore, $C_{n+2} = \{x_0\} \cup X$. Similarly, $C_{n+1} = \{y_0\} \cup Y$. Also, we have $x_j, y_j \notin C_i$, for every $1 \leqq i \neq j \leqq n$. Thus, $C_i = \{x_i, y_i\} \cup Z$, $1 \leqq i \leqq n$. Hence, the clique covering $\mathcal{C} = \{C_i : 1 \leqq i \leqq n+2\}$ is the unique clique covering of G_n with $n+2$ cliques and then $\mathrm{cc}(G_n) = n+2$. Consequently,

$$\mathrm{scc}'(G_n) = \sum_{C \in \mathcal{C}} |C| = n(n+2) + 2(n+1) = n^2 + 4n + 2.$$

On the other hand, the $n+4$ cliques $\{x_0\} \cup X$, $\{y_0\} \cup Y$, $X \cup Z$, $Y \cup Z$ and $\{x_i, y_i\}$, $1 \leqq i \leqq n$, form a clique covering $\mathcal{C}'$ and thus,

$$\mathrm{scc}(G_n) \leqq \sum_{C \in \mathcal{C}'} |C| = 2(n+1) + 2(2n) + 2n = 8n + 2.$$

Hence, the families of the optimum clique coverings achieving $\mathrm{cc}(G_n)$ and $\mathrm{scc}(G_n)$ are disjoint and $\mathrm{scc}'(G_n)/\,\mathrm{scc}(G_n)$ tends to infinity. $\square$

In the following, we prove some relations between $\mathrm{scc}(G)$, $\mathrm{scp}(G)$ and $\mathrm{cp}(G)$.

THEOREM 2. *If G is a graph with m edges and $\omega(G)$ is the clique number of G, then*

(i) $\frac{2m}{\omega(G)-1} \leqq \mathrm{scc}(G) \leqq \mathrm{scp}(G) \leqq 2m$,

(ii) $\frac{\mathrm{scp}^2(G)}{2m+\mathrm{scp}(G)} \leqq \mathrm{cp}(G)$.

Moreover, the first inequalities in (i) *and* (ii) *hold with equality, whenever the edge set of G can be partitioned into cliques of order $\omega(G)$ (in particular, when G is a triangle-free graph).*

PROOF. (i) Since the collection of all edges of G is a clique partition for G, we have $\mathrm{scc}(G) \leqq \mathrm{scp}(G) \leqq 2m$. Now, suppose that $\mathcal{C}$ is a clique covering of G such that $\sum_{C\in\mathcal{C}}|C| = \mathrm{scc}(G)$. Hence,

$$m \leqq \sum_{C\in\mathcal{C}} \binom{|C|}{2} \leqq \frac{\omega-1}{2}\sum_{C\in\mathcal{C}}|C| = \frac{1}{2}(\omega-1)\,\mathrm{scc}(G).$$

(ii) Let $\mathrm{cp}(G)=t$ and $\{C_1,\dots,C_t\}$ be a clique partition of G. Then, $m=\sum_{i=1}^{t}\binom{|C_i|}{2}$. Thus,

$$2m=\sum_{i=1}^{t}|C_i|^2-\sum_{i=1}^{t}|C_i| \geqq \frac{1}{t}\left(\sum_{i=1}^{t}|C_i|\right)^2-\sum_{i=1}^{t}|C_i| \geqq \frac{1}{t}\,\mathrm{scp}^2(G)-\mathrm{scp}(G),$$

where the second inequality is due to Cauchy–Schwarz inequality and the last inequality holds because the function $f(x)=\frac{1}{t}x^2-x$ is increasing for $x \geqq \frac{t}{2}$ and clearly $\mathrm{scp}(G) \geqq \mathrm{cp}(G)=t$.

Now assume that the edge set of G can be partitioned into l cliques of order $\omega(G)$. Thus, clearly $m=l\binom{\omega(G)}{2}$, $\mathrm{cp}(G)=l$ and $\mathrm{scp}(G)\leqq l\omega(G)=2m/(\omega(G)-1)$. Hence, $\mathrm{scp}(G)=l\omega(G)$ and the first inequalities in (i) and (ii) hold with equality. □

For a vertex $u\in V(G)$, let $N_G(u)$ denote the set of all neighbours of u in G and let $\overline{G}$ stand for the complement of G. Moreover, let $\Delta(G)$ be the maximum degree of G. Alon [1] proved that if G is a graph on n vertices and $\Delta(\overline{G})=d$, then $\mathrm{cc}(G)=O(d^2\log n)$. In the following, modifying the idea of Alon, we stablish an upper bound for $\mathrm{scc}(G)$.

THEOREM 3. *If G is a graph on n vertices with no isolated vertex and $\Delta(\overline{G})=d-1$, then*

$$\mathrm{scc}(G) \leqq (e^2+1)nd\left\lceil \ln\left(\frac{n-1}{d-1}\right)\right\rceil. \tag{1}$$

PROOF. Let p be a fixed number with $0<p<1$ and let S be a random subset of $V(G)$ defined by choosing every vertex u independently with probability p. For every vertex $u\in S$, if there exists a non-neighbour of u in S, then remove u from S. The resulting set is a clique of G. Repeat this procedure t times, independently, to get t cliques $C_1,C_2,\dots,C_t$ of G.

Let F be the set of all the edges which are not covered by the cliques $C_1,\dots,C_t$. For every edge uv, using inequality $(1-\alpha)\leqq e^{-\alpha}$, we have

$$\Pr(uv\in F)=\left(1-p^2(1-p)^{|N_{\overline{G}}(u)\cup N_{\overline{G}}(v)|}\right)^t$$

$$\leqq \left(1-p^2(1-p)^{2(d-1)}\right)^t \leqq e^{-tp^2(1-p)^{2(d-1)}}.$$

The cliques $C_1, \dots, C_t$ along with all edges in F comprise a clique covering of G. Hence,

$$\operatorname{scc}(G) \leqq \mathbf{E}\left(\sum_{i=1}^{t} |C_i| + 2|F|\right) \leqq npt + 2\binom{n}{2} e^{-tp^2(1-p)^{2(d-1)}}.$$

Now, set $p := 1/d$. Since $(1-1/d)^{d-1} \geqq 1/e$, we have

$$\operatorname{scc}(G) \leqq \frac{nt}{d} + n(n-1)e^{-td^{-2}e^{-2}}.$$

Finally, by setting $t := \left\lceil e^2 d^2 \ln\left(\frac{n-1}{d-1}\right)\right\rceil > 0$, we have

$$\operatorname{scc}(G) \leqq \frac{n(e^2d^2\ln(\frac{n-1}{d-1}) + 1)}{d} + n(d-1)$$

$$\leqq nd\left\lceil \ln\left(\frac{n-1}{d-1}\right)\right\rceil \left(e^2 + \frac{1}{\left\lceil \ln\left(\frac{n-1}{d-1}\right)\right\rceil}\right) \leqq nd\left\lceil \ln\left(\frac{n-1}{d-1}\right)\right\rceil (e^2+1). \quad \square$$

Now, one may naturally ask if the upper bound in (1) is tight. In other words, for every positive integers n and d, does there exist an n-vertex graph where the maximum degree of its complement is $d-1$ and its edge clique cover sum number is $\Omega\big(nd\log\left(\frac{n-1}{d-1}\right)\big)$? In the sequel, we are going to give an affirmative answer to this question at least for $d=2$.

A first candidate for graphs with large edge clique cover sum numbers is the family of complete multipartite graphs. The following theorem shows that when n is small in comparison to d, the upper bound is not met for the complete multipartite graphs. Nevertheless, we believe that it is the case when n is much larger than d. For positive integers n and k, an *orthogonal array* $\mathrm{OA}(n,k)$ is an $n^2 \times k$ array of elements in $\{1,\dots,n\}$, such that in every two columns each ordered pair (i,j), $1 \leqq i,j \leqq n$, appears exactly once.

THEOREM 4. *For positive integers n, t and d, let G be a complete t-partite graph on n vertices with at least two parts of size d and the other parts of size at most d. Then,* $\operatorname{scc}(G) \geqq nd$. *Moreover, if d is a prime power and $t \leqq d+1$, then* $\operatorname{scc}(G) = \operatorname{scp}(G) = nd$.

PROOF. Let $\mathcal{C}$ be a clique covering for G. For every vertex u, $N_G(u)$ contains a stable set (a set of pairwise nonadjacent vertices) of size d. Therefore, u is contained in at least d cliques of $\mathcal{C}$, i.e. the valency of u, $\mathcal{V}_{\mathcal{C}}(u)$ is at least d. Thus, $\operatorname{scc}(G) \geqq nd$.

Now, let d be a prime power. Since $t \leqq d+1$, there exists an orthogonal array $\mathrm{OA}(d,t)$. Denote the ith row of the orthogonal array by $a_{i1}, a_{i2}, \dots, a_{it}$ and let H be a complete t-partite graph on dt vertices with the parts

$V_1, \dots, V_t$, where $V_j = \{v_{j1}, \dots, v_{jd}\}$, for $1 \leqq j \leqq t$. For each $i \in \{1, \dots, d^2\}$, the set $C_i := \{v_{1a_{i1}}, v_{2a_{i2}}, \dots, v_{ta_{it}}\}$ is a clique of H. Since in every two columns of OA, each ordered pair (i, j), $1 \leqq i, j \leqq d$, appears exactly once, the collection $\mathcal{C} := \{C_i : 1 \leqq i \leqq d^2\}$ forms a clique partition for H. Moreover, for every vertex $u \in V(H)$, $\mathcal{V}_{\mathcal{C}}(u) = d$. On the other hand, G is an induced subgraph of H. Thus, the collection $\mathcal{C}' := \{C_i \cap V(G) : 1 \leqq i \leqq d^2\}$ is a clique partition of G and for every vertex $u \in V(G)$, $\mathcal{V}_{\mathcal{C}'}(u)$ is at most d. Hence, $\operatorname{scc}(G) \leqq \operatorname{scp}(G) \leqq nd$. $\square$

For positive integers t and d, let us denote the complete t-partite graph on td vertices, each part of size d by $K_t(d)$. Theorem 3 asserts that $\operatorname{scc}\big(K_t(d)\big) \leqq cd^2 t \log t$, for some constant c. Although Theorem 4 says that $\operatorname{scc}\big(K_t(d)\big) = d^2 t$ when $t \leqq (d+1)$ and d is a prime power, we believe that $\operatorname{scc}\big(K_t(d)\big) = \Omega(d^2 t \log t)$, when d is constant and t is sufficiently large. This leads us to the following conjecture.

Conjecture 5. There exists a function f and a constant c, such that for every positive integers t and d, if $t \geqq f(d)$, then $\operatorname{scc}\big(K_t(d)\big) \geqq cd^2 t \log t$.

In fact, if Conjecture 5 is correct, then the upper bound in (1) is best possible up to a constant factor. In the following section, we will prove that Conjecture 5 is true for $d = 2$.

3. Cocktail party graphs

In this section, we investigate the edge clique cover sum number of the cocktail party graph $K_t(2)$. Given a positive integer t, the cocktail party graph $K_t(2)$ is obtained from the complete graph K_{2t} with the vertex set $\{x_1, \dots, x_t\} \cup \{y_1, \dots, y_t\}$ by removing all the edges $x_i y_i$, $1 \leqq i \leqq t$.

Various clique covering parameters of the cocktail party graphs have been studied in the literature. Orlin [14] asked about the asymptotic behaviour of $\operatorname{cc}\big(K_t(2)\big)$, with the motivation that it arises in an optimization problem in Boolean functions theory. He also conjectured that $\operatorname{cp}\big(K_t(2)\big) \sim t$. Gregory et al. [9] proved that for $t \geqq 4$, $\operatorname{cp}\big(K_t(2)\big) \geqq 2t$ and for large enough t, $\operatorname{cp}\big(K_t(2)\big) \leqq 2t \log\log 2t$. The problem that $\operatorname{cp}\big(K_t(2)\big) \sim 2t$ is still an open problem. Moreover, Gregory and Pullman [10], by applying a Sperner-type theorem of Bollobás and Schönheim on set systems, proved that for every integer t, $\operatorname{cc}\big(K_t(2)\big) = \sigma(t)$, where

$$\sigma(t) = \min\left\{k : t \leqq \binom{k-1}{\lceil k/2 \rceil}\right\}.$$

Furthermore, the authors in [5], using the pairwise balanced designs, have proved that $\operatorname{scp}\big(K_t(2)\big) \sim (2t)^{3/2}$.

Here, using the following well-known theorem by Bollobás, we prove a lower bound for the edge clique cover sum number of $K_t(2)$ which determines the asymptotic behaviour of $\operatorname{scc}\big(K_t(2)\big)$ and implies that Conjecture 5 is true for $d=2$.

BOLLOBÁS' TWO FAMILIES THEOREM [2]. *Let $A_1,\dots,A_t$ be some sets of size $a_1,\dots,a_t$, respectively and $B_1,\dots,B_t$ be some sets of size $b_1,\dots,b_t$, respectively, such that $A_i\cap B_j=\emptyset$ if and only if $i=j$. Then*

$$\sum_{i=1}^{t}\binom{a_i+b_i}{a_i}^{-1}\leqq 1. \tag{2}$$

THEOREM 6. *Let $K_t(2)$ be the cocktail party graph on $2t$ vertices. Then*

$$t\rho(t)\leqq \operatorname{scc}\big(K_t(2)\big)\leqq t\sigma(t),$$

where $\sigma(t)$ is defined as above and $\rho(t)=\min\Big\{k-1: t\leqq\binom{k}{\lceil k/2\rceil}\Big\}$.

PROOF. Since $\operatorname{cc}\big(K_t(2)\big)=\sigma(t)$ and every clique in $K_t(2)$ is of size at most t, we have $\operatorname{scc}\big(K_t(2)\big)\leqq t\sigma(t)$. For the lower bound, assume that $\{C_1,\dots,C_k\}$ is an arbitrary clique covering for $K_t(2)$. For every $i\in\{1,\dots,t\}$, define

$$A_i=\{a: x_i\in C_a\},\quad B_i=\{a: y_i\in C_a\}.$$

Also, let $a_i=|A_i|$, $b_i=|B_i|$ and $c_i=a_i+b_i$. Then for every $i\neq j$, there exists a clique containing the edge x_iy_j. Hence, $A_i\cap B_j\neq\emptyset$. Moreover, since no clique contains both vertices x_i and y_i, we have $A_i\cap B_i=\emptyset$. Therefore, by Bollobás' theorem, we have (2).

For every integer m, let $f(m)=\binom{m}{\lceil m/2\rceil}^{-1}$ and $f(x)$ be the linear extension of $f(m)$ in $\mathbb{R}^+$. Since f is non-increasing and convex, by Jensen inequality, we have

$$f\left(\left\lceil\frac{1}{t}\sum_{i=1}^{t}c_i\right\rceil\right)\leqq f\left(\frac{1}{t}\sum_{i=1}^{t}c_i\right)\leqq\frac{1}{t}\sum_{i=1}^{t}\binom{c_i}{\lceil c_i/2\rceil}^{-1}\leqq\frac{1}{t}\sum_{i=1}^{t}\binom{a_i+b_i}{a_i}^{-1}\leqq\frac{1}{t}.$$

Thus, $\binom{\lceil\frac{1}{t}\sum_{i=1}^{t}c_i\rceil}{\lceil\frac{1}{2t}\sum_{i=1}^{t}c_i\rceil}\geqq t$. Therefore,

$$\rho(t)\leqq\left\lceil\frac{1}{t}\sum_{i=1}^{t}c_i\right\rceil-1\leqq\frac{1}{t}\sum_{i=1}^{t}c_i=\frac{1}{t}\sum_{a=1}^{k}|C_a|.$$

Consequently, $t\rho(t) \leqq \operatorname{scc}\big(K_t(2)\big)$. $\square$

Theorem 6 along with the approximation $\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}$ yields the following corollary which proves Conjecture 5 for $d = 2$.

Corollary 7. *For every integer* t, $\operatorname{scc}\big(K_t(2)\big) \sim t \log t$.

4. Concluding remarks

In the previous section, by considering a clique covering as a set system and applying Bollobás' theorem, we proved Conjecture 5 for $d = 2$. From this point of view, the conjecture can be restated as an interesting set system problem and thus it can be viewed as a generalization of Bollobás' two families theorem, as follows. For other multipartite versions of Bollobás' inequality, see a two-part survey by Tuza [18,19].

Conjecture 8. Let $d \geqq 2$, $t \geqq 1$ and $\mathcal{F} = \big\{\big(A_i^1, A_i^2, \dots, A_i^d\big) : 1 \leqq i \leqq t\big\}$ such that A_i^j is a set of size k_{ij} and $A_i^j \cap A_{i'}^{j'} = \emptyset$ if and only if $i = i'$ and $j \neq j'$. Then, there exists a function f and a constant c, such that for every $t \geqq f(d)$,

$$\sum_{i,j} k_{ij} \geqq c d^2 t \log t.$$

It is worth noting that Conjecture 8 is equivalent to Conjecture 5 and thus its correctness would imply that the upper bound in (1) is tight up to a constant factor.

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