On incidence coloring for some graphs of maximum degree 4

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Abstract

Brualdi and Massey in 1993 posed two conjectures regarding the upper bound for incidence coloring number of graphs in terms of maximum degree. In this paper among some results, we prove these conjectures for some classes of graphs with maximum degree 4.

Keywords: Incidence coloring; Incidence coloring number; Incidence coloring conjecture.

1 Introduction

Throughout the paper, $G = (V, E)$ is a finite, undirected and simple graph of order $n(G)$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For the definitions and notations, we follow [4].

Suppose that v is an arbitrary vertex in G and e is an edge incident to v. The pair (v, e) is called an *incidence* in G. The set of all of incidences in G is denoted by I(G);

$$I(G) = \{(v, e) \in V(G) \times E(G) : \text{edge } e \text{ is incident to } v\}.$$

Two incidences (v, e) and (w, f) are said to be adjacent in G if one of the following conditions holds:
i) $v = w$;

ii) $e = f$;

iii) the edge vw equals to e or f.

The *incidence coloring* of G is a mapping $\sigma : I(G) \rightarrow S$ (S is the color set) in which adjacent incidences receive different colors. If $|S| = k$, then G is said *k-incidence colorable.* The minimum k for which G is k-incidence colorable, is the *incidence coloring number* of G denoted by $\chi_i(G)$. For a subgraph H of G, we mean by $S(H)$ the colors used for the incidences of H in coloring σ.

In what follows, for convenience "coloring" means "incidence coloring". For an arbitrary vertex u of $V(G)$, we denote the set of all incidences of form (u, uv) by $I^+(u)$ and the set of all incidences of form (v, vu) by $I^-(u)$ and we show each incidence (u, uv) by (uv). If $\sigma(uv) = i$, then the phrase "affected incidences from color i", refers to all of the incidences which are adjacent to the incidence (uv). For a subgraph H of G, $I_H^-(u)$ means the set of incidences of $I^-(u)$ which belong to $I(H)$. Also, $N_H(u)$ means the set of neighbors of u in H.

The concept of incidence coloring was introduced by Brualdi and Massey in 1993 [5]. They proved that in general:

$$\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G) \tag{1}$$

The lower bound for some graphs such as complete graphs and trees; also the upper bound for cycles of order not divisible by 3 are attained. Among some other results in [5], the following two conjectures, which are equivalent for $\Delta(G) \leq 3$, are posed.

Conjecture 1. *The upper bound is never attained for graphs with $\Delta(G) > 2$.*

Conjecture 2. *For every graph G, $\chi_i(G) \leq \Delta(G) + 2$.*

Conjecture 2 is known as the incidence coloring conjecture (ICC).

The ICC is proved for some graphs such as complete graphs, trees, complete bipartite graphs [5], graphs with $\Delta(G) = 3$ [9], K_4-minor free graphs [8], outerplanar 2-connected graphs [11].

In [7] Guiduli showed that the concept of incidence coloring is a special case of directed star arboricity introduced by Alon and Algor in [1]. Following their work, he showed that the Paley graph of order p with $p \equiv 1 \pmod 4$ is an example to prove that ICC is false. Moreover, he proved that $\chi_i(G) \geq \Delta + \Omega(\log(\Delta))$, where $\Omega = 1/8 - o(1)$. However this asymptotic bound left open the possibility of the truth of ICC for graphs with small maximum degree. The ICC for the cases $\Delta(G) = 1$ and 2 is straightforward. In [10] Shiu et al, proved the ICC for cubic Hamiltonian graphs and some other cubic graphs and they conjectured truth of ICC in general case for cubic graphs. In [9] Maydanskiy proved ICC for all graphs with maximum degree 3. He conjectured that ICC would be true for the graphs of maximum degree 4.

In this paper, we will investigate the incidence coloring of some graphs of maximum degree 4. According to the upper bound in (1), for such graphs, $\chi_i(G) \leq 8$. Regarding to prove the first conjecture, we decrease this bound to 7 for some 4-regular graphs. Also, we prove the ICC for some classes of graphs with maximum degree 4.

The following proposition is easy and its proof is omitted.

Proposition 1. *If G is an edge disjoint union of two subgraphs G_1 and G_2, then $\chi_i(G) \leq \chi_i(G_1) + \chi_i(G_2)$.*

In [6] it is proved that for every cycle C_n of order divisible by 3, $\chi_i(C_n) = 3$ and for the other cycles, $\chi_i(C_n) = 4$. Also it is proved that for every path P_n, $\chi_i(P_n) = 3$. For convenience in our expressions because of the repetition, we give a coloring of cycles and paths and fix this coloring throughout the paper.
Let $C_n : v_1v_2...v_nv_1$ be a cycle of order n, $v_i \in V(C_n)$ and $u \in N_{C_n}(v_i)$. We define the incidence colorings σ_0, σ_1 and σ_2 as follows.

I) If $n = 3k$, let $\sigma_0 : I(C_n) \to \{1,2,3\}$, where for every i, $1 \leq i \leq n$, $\sigma_0(uv_i) =$
$$\begin{cases} 3 & i \equiv 0 \pmod 3 \\ 2 & i \equiv 1 \pmod 3 \\ 1 & i \equiv 2 \pmod 3 \end{cases}$$
II) If $n = 3k+1$, let $\sigma_1 : I(C_n) \to \{1,2,3,4\}$, where $\sigma_1(I^-(v_1)) = 4$ and for every i, $2 \leq i \leq n$,

$$\sigma_1(uv_i) = \begin{cases} 3 & i \equiv 0 \pmod 3 \\ 2 & i \equiv 1 \pmod 3 \\ 1 & i \equiv 2 \pmod 3 \end{cases}$$

III) If $n = 3k+2$, let $\sigma_2 : I(C_n) \to \{1, 2, 3, 4\}$, where $\sigma_2(v_2v_1) = 2$, $\sigma_2(v_nv_1) = 3$, $\sigma_2(v_{n-1}v_n) = 1$, $\sigma_2(v_1v_n) = 4$, and for i, $2 \le i \le n-1$,

$$\sigma_2(uv_i) = \begin{cases} 3 & i \equiv 0 \pmod 3 \\ 2 & i \equiv 1 \pmod 3 \\ 1 & i \equiv 2 \pmod 3 \end{cases}$$

Note that if $P_n : v_1v_2...v_n$ be a path, then the restriction of σ_0 on $I(P_n)$ is an incidence coloring of P_n by 3 colors. Later on, we denote this coloring by σ'.

2 The truth of Conjecture 1 for some graphs with $\Delta = 4$

In this section, we prove that Conjecture 1 is true for some classes of 4-regular graphs. A 2-factor in a graph G is a 2-regular spanning subgraph of G. It is known that every 4-regular graph is 2-factorable, that means $E(G)$ is disjoint union of two 2-factors [4].

Theorem 1. *Suppose that G is a 4-regular graph with a cycle decomposition into two 2-factors F_1 and F_2.*
I) If the orders of all components of F_1 are divisible by 3 except exactly one cycle which is of order $3k+1$ and F_2 is C_4-free, then $\chi_i(G) \le 7$.
II) If the orders of all components of F_1 are divisible by 3 except exactly one cycle which is of order $3k+2$, then $\chi_i(G) \le 7$.

Proof. I) Let $F_1 = C_1 \cup C_2 \cup ... \cup C_t$, where $C_1 : v_1v_2...v_lv_1$, $n(C_1) = 3k+1$ and $n(C_i) \equiv 0 \pmod 3$, $2 \le i \le t$. First we color the cycles C_i, $2 \le i \le t$, as coloring σ_0 and the cycle C_1 as coloring σ_1. Then, we color 2-factor F_2 as follows. First set $\sigma(I^-_{F_2}(v_1)) = 4$. Now, we color the affected incidences from color 4. Clearly afterwards, we can color the remaining incidence of F_2 by four colors $\{4, 5, 6, 7\}$.

Let $N_{F_2}(v_2) = \{v_{k_1}, v_{k_2}\}$ and $N_{F_2}(v_l) = \{v_{k_3}, v_{k_4}\}$. We call the sections $v_{k_1}v_2v_{k_2}$ and $v_{k_3}v_lv_{k_4}$ as *subpaths*. Since by assumption F_2 is C_4-free, vertices v_2

and v_l have at most one common neighbor. If $N_{F_2}(v_2) \cap N_{F_2}(v_l) = \varnothing$, then the subpaths $v_{k_1}v_2v_{k_2}$ and $v_{k_3}v_lv_{k_4}$ are colorable as σ' separately using the colors $\{5, 6, 7\}$. If $|N_{F_2}(v_2) \cap N_{F_2}(v_l)| = 1$, say $v_{k_2} = v_{k_3} \in N_{F_2}(v_2) \cap N_{F_2}(v_l)$, then the subpath $v_{k_1}v_2v_{k_2}v_lv_{k_4}$ is colorable as σ' using the colors $\{5, 6, 7\}$. Obviously after this, the affected incidences from color 4 on the vertices v_2 and v_l are colored.

Let $N_{F_2}(v_1) = \{v_{i_1}, v_{i_2}\}$. We assign $\sigma(I^-_{F_2}(v_{i_1})) = 5$ and $\sigma(I^-_{F_2}(v_{i_2})) = 6$. Now we consider the incidences affected from 4 by the vertices v_{i_1} and v_{i_2}. Suppose $N_{F_2}(v_{i_1}) = \{v_1, v_{t_1}\}$ and $N_{F_2}(v_{i_2}) = \{v_1, v_{t_2}\}$ (since F_2 is C_4-free, $t_1 \neq t_2$). According to described coloring until now, to obtain a coloring, we must have $\sigma(v_{i_1}v_{t_1}) \in \{6, 7\}$ and $\sigma(v_{i_2}v_{t_2}) \in \{5, 7\}$.

Now the remaining uncolored incidences of F_2 are union of some paths. To complete the coloring, we can color these incidences using the colors $\{4, 5, 6, 7\}$. Since these incidences are not affected by incidences with color 4 and the colors used on incidences in F_1 are $\{1, 2, 3\}$, such a coloring is possible (in Figure 1(I) an example of such coloring is illustrated. 2-factor F_1 is shown by black lines and 2-factor F_2 is shown by dash lines).

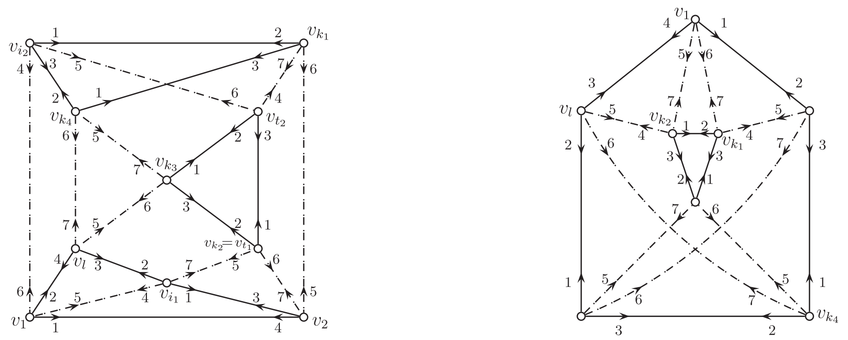
II) Let $F_1 = C_1 \cup C_2 \cup ... \cup C_t$, where $C_1 = v_1v_2v_3...v_lv_1$, $n(C_1) = 3k + 2$ and $n(C_i) \equiv 0 \pmod 3$, $2 \leq i \leq t$. First we color the cycles C_i as coloring σ_0 and the cycle C_1 as coloring σ_2. Now we provide a coloring on the incidences of F_2.

For this purpose, first we color the affected incidences by color 4 and afterwards color the remaining incidences using colors $\{4, 5, 6, 7\}$. Suppose that $N_{F_2}(v_1) = \{v_{k_1}, v_{k_2}\}$ and $N_{F_2}(v_l) = \{v_{k_3}, v_{k_4}\}$. If $N_{F_2}(v_1) \bigcap N_{F_2}(v_l) = \varnothing$, then color the subpaths $v_{k_1}v_1v_{k_2}$ and $v_{k_3}v_lv_{k_4}$ by $\{5, 6, 7\}$ such that $\sigma(I^-_{F_2}(v_1)) = \sigma(I^-_{F_2}(v_l)) = 7$. If $|N_{F_2}(v_1) \bigcap N_{F_2}(v_l)| = 1$, say $v_{k_2} = v_{k_3} \in N_{F_2}(v_1) \bigcap N_{F_2}(v_l)$, then color the subpath $v_{k_1}v_1v_{k_2}v_lv_{k_4}$ by $\{5, 6, 7\}$. If $|N_{F_2}(v_1) \bigcap N_{F_2}(v_l)| = 2$, then $v_{k_1}v_1v_{k_2}v_lv_{k_1}$ forms a 4-cycle C'. To obtain a coloring, set:

$$\sigma(I^-_{F_2}(v_1)) = 7, \sigma(I^-_{F_2}(v_l)) = 4, \sigma(I^-_{F_2}(v_{k_1})) = 6, \sigma(I^-_{F_2}(v_{k_2})) = 5$$

The assignment σ yields a coloring of C'.
So far, we have colored all of the incidences affected by color 4. Similar to the previous part, the remaining incidences in F_2 are affected by three colors $\{1, 2, 3\}$ and those can be colored using four colors $\{4, 5, 6, 7\}$ (Figure 1(II) is an example of such coloring. 2-factor F_1 is shown by black lines and 2-factor F_2 is shown by

dash lines). ■



(I) (II)

Figure 1: An example for Theorem 1.

Theorem 2. *If G is a 4-regular graph decomposed into two Hamiltonian cycles, then $\chi_i(G) \leq 7$.*

Proof. Suppose that $C_1 : v_1 \overbrace{v_2 \ldots v_n}^{P_1} v_1$ and $C_2 : v_1 \overbrace{v_j v_{j_1} \ldots v_{j_{n-2}}}^{P_2} v_1$, $3 \leq j \leq n-1$, are two Hamiltonian cycles of G.

We proceed to give a coloring of G. We color the subpath P_1 as coloring σ', such that $S(P_1) = \{1, 2, 3\}$. Similarly, we color the subpath P_2 as coloring σ', such that $S(P_2) = \{4, 5, 6\}$. Let $v \in V(G) - (\{v_1\} \cup N(v_1))$, $N_{C_1}(v) = \{x_1, x_2\}$ and $N_{C_2}(v) = \{y_1, y_2\}$. According to the coloring, for colored incidences, we have $\sigma'(vx_i) \neq \sigma'(vx_j) \neq \sigma'(x_i v) \in \{1, 2, 3\}$ and $\sigma'(vy_i) \neq \sigma'(vy_j) \neq \sigma'(y_i v) \in \{4, 5, 6\}$, for $i, j = 1, 2$. Also, $\sigma'(vx_i) \neq \sigma'(vy_i)$ and $\sigma'(x_i v) \neq \sigma'(vy_i)$. Hence, the affected incidences in an arbitrary vertex v have received different colors.

To complete the coloring of G it suffices to offer proper colors for the incidences in $I^+(v_1)$ and $I^-(v_1)$. For this sake, first we color the incidences of $I^+(v_1)$. Since $\sigma(v_2 v_3) = 1$, choose $a_1 \in \{2, 3\}$ and $a_2 \in \{1, 2, 3\} - \{a_1, \sigma'(v_n v_{n-1})\}$ and assign $\sigma(v_1 v_2) = a_1$ and $\sigma(v_1 v_n) = a_2$. Similarly, there are two permitted colors in $\{4, 5, 6\}$ to be assigned for the incidences $(v_1 v_j)$ and $(v_1 v_{j_{n-2}})$.

To complete the coloring of G and according to this fact that the incidences in $I^-(v_1)$ are not adjacent, we assign $\sigma(v_2v_1)=\sigma(v_nv_1)=\sigma(v_jv_1)=\sigma(v_{j_{n-2}}v_1)=7$. Therefore, $\chi_i(G)\leq 7$ (in Figure 2 an example of such coloring is illustrated. Two Hamiltonian cycles are discriminated by black and dash lines). ■

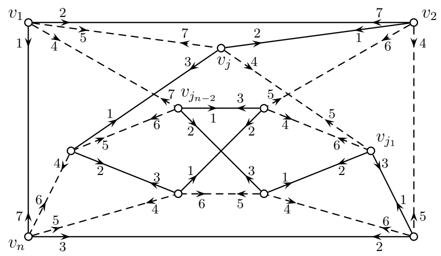


Figure 2: An example for Theorem 2.

The existence of two edge disjoint Hamiltonian cycles in some 4-regular graphs has been proved. In the following, we introduce some classes of such graphs and conclude that Conjecture 1 is true for these graphs.

Definition 1. [4] *Let Γ be a finite abelian group and S be a set of elements of Γ not including the identity element. Suppose, furthermore, that the inverse of every element of S also belongs to S. The Cayley graph of Γ with respect to S is the graph $CG(\Gamma, S)$ with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1}\in S$.*

Theorem 3. [3] Every 4-regular Cayley graph contains two edge disjoint Hamiltonian cycles.

Corollary 1. *If G is a 4-regular Cayley graph, then $\chi_i(G)\leq 7$.*

Definition 2. [2] *The Butterfly graph of dimension n, denoted by $BF(n)$, is the graph with vertex set $\mathbb{Z}_n\times\mathbb{Z}_2^n$ and with edges defined as follows. Any vertex (l,x), where $l\in\{0,...,n-1\}$, $x=x_0x_1...x_{n-1}, x_i\in\mathbb{Z}_2, 0\leq i\leq n-1$, is adjacent to the vertex $(l+1,x)$ and to the vertex $(l+1,x(l))$, where $x(l)=x_0x_1...x_{l-1}\bar{x}_lx_{l+1}...x_{n-1}$.*

Theorem 4. [2] *The Butterfly graph contains two edge disjoint Hamiltonian cycles.*

According to 4-regularity of Butterfly graph, Theorem 2 ensures the truth of Conjecture 1 for this graph.

Corollary 2. *If G is a Butterfly graph, then $\chi_i(G) \leq 7$.*

3 The truth of ICC for some graphs with $\Delta = 4$

In this Section, we prove that ICC (Conjecture 2) is true for some classes of graphs of maximum degree 4.

Theorem 5. *Suppose that G is a 4-regular graph decomposed into two 2-factors F_1 and F_2. If the order of every component of F_1 and F_2 is divisible by 3, then $\chi_i(G) \leq 6$.*

Proof. Let $F_1 = C_{11} \bigcup C_{21} \bigcup ... \bigcup C_{r1}$ and $F_2 = C_{12} \bigcup C_{22} \bigcup ... \bigcup C_{t2}$, where $n(C_{i1}) \equiv 0 \pmod 3$, $n(C_{j2}) \equiv 0 \pmod 3$, $1 \leq i \leq r, 1 \leq j \leq t$. Since $\chi_i(C_{i1}) = \chi_i(C_{j1}) = 3$, $1 \leq i \leq r, 1 \leq j \leq t$, by Proposition 1 we get, $\chi_i(G) \leq \chi_i(F_1) + \chi_i(F_2) = 6$ (in Figure 3 an example of such coloring is illustrated). ■

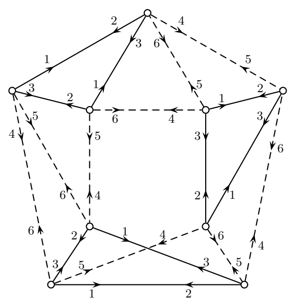


Figure 3: An example for Theorem 5.

Corollary 3. *Let G be a 4-regular graph of order divisible by 3 which can be decomposed into two Hamiltonian cycles. Then, $\chi_i(G) \leq 6$*

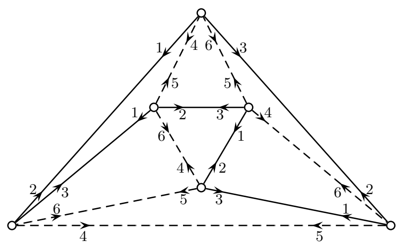


Figure 4: An example for Corollary 3.

It is known that a $2k$-regular graph consists of k edge disjoint 2-factors [4]. According to this fact, we have the following corollary.

Corollary 4. *Let G be a $2k$-regular graph. If $E(G) = F_1 \cup F_2 \cup ... \cup F_k$ where the order of each component in every 2-factor F_i is divisible by 3, then G is $3k$-incidence colorable.*

Proof. Every 2-factor F_i, $1 \leq i \leq k$, is colorable with 3 colors (among $3k$ colors). Therefore by Proposition 1, $\chi_i(G) \leq 3k$. ■

Theorem 6. *If G is a graph with maximum degree 4 which has a decomposition into two Hamiltonian paths, then $\chi_i(G) \leq 6$.*

Proof. Let $G = P_1 \cup P_2$, where P_1 and P_2 are two Hamiltonian paths. We color P_1 as coloring σ' such that, $S(P_1) = \{1, 2, 3\}$ and path P_2 as coloring σ' such that, $S(P_2) = \{4, 5, 6\}$. This assignment gives an incidence coloring of G. Thus by Proposition 1, $\chi_i(G) \leq \chi_i(P_1) + \chi_i(P_2) = 3 + 3 = 6$ (in Figure 5 an example of such coloring is illustrated). ■

Corollary 5. *Let G be a 4-regular graph decomposed into 2-factors F_1 and F_2, where F_1 is a Hamiltonian cycle and every component of F_2 is a cycle of order multiple of 3 except one cycle C of order $3k + 2$. If e is an edge in C, then $\chi_i(G - e) \leq 6$.*

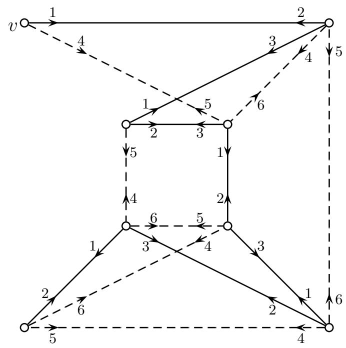


Figure 5: An example for Theorem 6.

Proof. According to coloring σ', $P = C - e$ is 3-incidence colorable. If $e = uv$, then there exists $a \in \{1, 2, 3\}$ such that $a \notin I^+_P(u) \cup I^-_P(u)$. By the assumption G is of order $n \equiv 2 \pmod 3$. Let $F_1 : uv_1v_2...v_{n-1}u$. We color F_1 as coloring σ_2 such that $S(F_1) = \{4, 5, 6, a\}$ and $\sigma_2(uv_{n-1}) = a$. Finally, we color the cycle components of order $3k$ in F_2 as coloring σ_0 by colors $\{1, 2, 3\}$, by a renaming if necessary, such that $I^-_{F_2}(v_{n-1}) = \{a\}$. ■

Theorem 7. *Suppose that G is a graph of maximum degree 4 decomposed into a Hamiltonian cycle C and a Hamiltonian path P.*

(I) *Let $n(G) = 3k + 1$ and $C : \overbrace{v_1v_2...v_t}^{\tilde{C_1}}...v_nv_1$. If $P : v_1v_3v_j...v_tv_2v_l...$, where $n(\tilde{C_1}) \equiv 1 \pmod 3$, then $\chi_i(G) \leq 6$.*

(II) *Let $n(G) = 3k + 2$. If $C : v_1v_2...v_3v_nv_1$ and $P : v_1v_{j_1}v_{j_2}...v_{j_k}...v_l$, where $v_{j_k} = v_n$ for some $j_k \equiv 0 \pmod 3$, then $\chi_i(G) \leq 6$.*

Proof. (I) First we provide a coloring of C by colors $\{1, 2, 3, 4\}$. In this case, we consider σ_1 as the following:

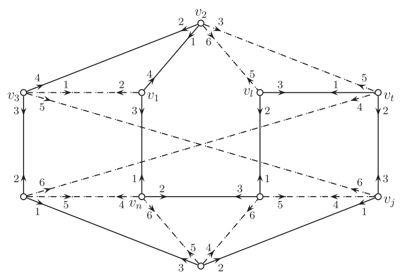
$\sigma_1(I^-_C(v_1)) = 1$, $\sigma_1(I^-_C(v_2)) = 4$, and for every i, $3 \leq i \leq n$, $\sigma_1(uv_i) =$

$$\begin{cases} 2 & i \equiv 0 \pmod 3 \\ 3 & i \equiv 1 \pmod 3 \\ 1 & i \equiv 2 \pmod 3 \end{cases}$$

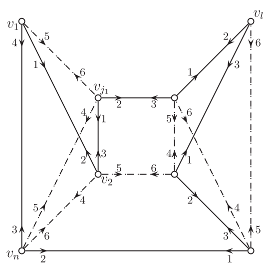
Now we color the incidences in P. First, we consider the incidences of P

affected from color 4 on v_1 and v_3. Since $\sigma(I_C^-(v_3)) = 2$ and $\sigma(I_C^+(v_3)) = \sigma(I_C^+(v_1)) = \{3,4\}$, we can set $\sigma(v_3v_1) = 1$. Similarly, since $\sigma(I_C^-(v_1)) = 1$ and $\sigma(I_C^+(v_1)) = \sigma(I_C^+(v_3)) = \{3,4\}$, we set $\sigma(v_1v_3) = 2$. Now we color subpath $P_1 : v_3v_j...v_t$ starting from the incidence v_3v_j as coloring σ' by renaming color 1 to 5, 2 to 6, and 3 to 4. Also, subpath $P - v_1v_3v_j...v_tv_2v_l$ is colorable as σ' by colors $\{4,5,6\}$. Now to complete the coloring it is sufficient to color incidences on v_2 in P which are affected from color 4. Note that $\sigma(I_C^-(v_2)) = 4$ and $\sigma(I_C^+(v_2)) = \{1,2\}$. Also, Since $n(\tilde{C}_1) \equiv 1 \pmod 3$, it can easily be checked that $\sigma(I_C^-(v_t)) = 3$. Thus, we assign $\sigma(v_2v_t) = 3$. Now it suffices to set $\sigma(I_P^-(v_2)) = 5$ and $\sigma(v_2v_l) = 6$. With this assignment the affected incidences of the color 4 on v_2 have received different colors and the coloring is completed (Figure 6(I) is an example of such coloring. The cycle and path are discriminated by black and dash lines).

(II) First we color the cycle C as σ_2. Now it suffices to color P in such a way that adjacent incidences receive different colors. Consider the subpath $P_1 : v_1v_{j_1}v_{j_2}...v_{j_k}$. We color P_1 as coloring σ', by renaming color 1 to 5, 2 to 6 and 3 to 4. Note that $4 \in I_C^-(v_n)$ and due to the assumption $n = j_k$, for some $j_k \equiv 0 \pmod 3$, it is easily checked that $4 \notin I_P^+(v_n)$. Hence, $4 \in I_P^-(v_n)$. With this assignment, the remaining incidences of P are colorable by colors $\{4,5,6\}$ and the coloring of G with 6 colors is completed (Figure 6(II) is an example of such coloring. The cycle and path are discriminated by black and dash lines). ■



(I)



(II)

Figure 6: An example for Theorem 7.

4 Conclusion

In Section 2, Conjecture 1 is proved for 4-regular graphs which are decomposed into two Hamiltonian cycles (Theorem 2). Also, the ICC is proved for graphs of maximum degree 4 which are decomposed into two Hamiltonian paths (Theorem 6). Thus, if G is a 4-regular graph containing two disjoint Hamiltonian cycles, then by deleting one edge from each Hamiltonian cycle, we obtain a subgraph satisfying the ICC. Moreover, in Section 3, with some assumptions for graphs of maximum degree 4 decomposed into a Hamiltonian cycle and a Hamiltonian path ICC is proved.

Furthermore, regarding to investigating the truth of ICC, we have considered an equivalent definition of incidence coloring as follows.

Suppose $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{e_1, ..., e_m\}$. The *incidence matrix* of G is defined to be the $n \times m$ matrix B, (each row indicates a vertex and each column indicates an edge), such that

$$B = (b_{ij}) = \begin{cases} 1 & \text{the vertex } v_i \text{ is incident to the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

Thus, every entry 1 in B indicates one of the elements of I(G). An incidence coloring of G is a labeling of the entries 1 in B such that:
i) No two 1's in the same row receive the same label;
ii) No two 1's in the same column receive the same label;
iii) If the labels of the entries ij and kl in B are equal, then $b_{il} \neq 1$ and $b_{kj} \neq 1$.

Using this definition, we have provided a computer program to determine the incidence coloring number of 4-regular graphs. This program for all 4-regular graphs of order at most 12 gives an incidence coloring by 6 colors.

These results are getting closer to prove ICC in general. Therefore, attempting to prove ICC for graphs with $\Delta = 4$ would be worthwhile.

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