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RESEARCH PAPER

Star Edge Coloring of Cactus Graphs

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Abstract

A star edge coloring of a graph G is a proper edge coloring of G such that no path or cycle of length 4 is bicolored. The star chromatic index of G, denoted by $\chi'_s(G)$, is the minimum k such that G admits a star edge coloring with k colors. Bezegová et al. (J Graph Theory 81(1):73–82, 2016) conjectured that the star chromatic index of outerplanar graphs with maximum degree Δ is at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$. In this paper, we prove this conjecture for a class of outerplanar graphs, namely Cactus graphs, wherein every edge belongs to at most one cycle.

Keywords Star edge coloring · Star chromatic index · Outerplanar graphs · Cactus graphs

Mathematics Subject Classification 05C15

1 Introduction

All graphs in this paper are simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We show every edge $e \in E(G)$ with two endpoints $u, v \in V(G)$, by $e = uv$, and say that u and v are adjacent. Moreover, we say that two edges e and e' are *adjacent* if they have a common endpoint.

A *proper edge coloring* of G is an assignment of colors to its edges such that no two adjacent edges have the same color. There are variants coloring of graphs under the additional constraints. For example, *star edge coloring* is a kind of proper edge coloring with no bicolored path or cycle of length 4 (path or cycle with four edges). The *star chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum k such that G has a star edge coloring with k colors (Dvořák et al. 2013; Liu and Deng 2008). Star edge coloring was defined in 2008 by Liu and Deng (2008).

Bezegová et al. (2016) obtained upper bound $\left\lfloor \frac{3\Delta}{2} \right\rfloor$ for the star chromatic index of trees. Omoomi et al. (2018) gave a polynomial time algorithm to find the star chromatic index of every tree. Using the star chromatic index of trees, Bezegová et al. (2016) found upper bound $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 12$ for the star chromatic index of outerplanar graphs. An *outerplanar* graph is a graph that has a planar drawing for which all vertices belong to the outer face. Bezegová et al. also presented the following conjecture.

Conjecture 1 Bezegová et al. (2016) *If G is an outerplanar graph with maximum degree Δ, then*

$$\chi'_s(G) \le \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1.$$

Wang et al. (2018) proved that $\chi'_s(G) \le \left\lfloor \frac{3\Delta}{2} \right\rfloor + 5$ for outerplanar graph G with maximum degree Δ.

A *Cactus* is a graph in which every edge belongs to at most one cycle. Since these graphs are outerplanar, in order to prove Conjecture 1, it is worth to study the star edge coloring of Cactus graphs. In this paper, we prove Conjecture 1 for Cactus graphs with maximum degree Δ.

A *unicyclic Cactus* (or UCC for short) is a Cactus $G = C \cup F$, where C is a cycle (or an edge) and F is a forest consisting of some rooted trees with height at most two and the roots in C (Fig. 1). The *height* of a rooted tree is the length of the longest path between the root and a leaf. We denote a rooted tree with root v as T_v.

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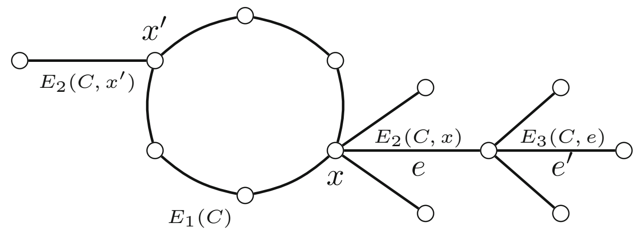

Fig. 1 A UCC graph with three types of edges

Through the paper, we assume that the degree of all vertices except leaves is Δ, because adding some leaves to the vertices with degree at least two does not reduce the star chromatic index. We call such a graph a Δ-*semiregular*.

The structure of the paper is as follows. In Sect. 2, we present an algorithm to give a star edge coloring for every Δ-semiregular UCC with at most $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors. Using this coloring, in Sect. 3, we find a star edge coloring for every Δ-semiregular Cactus with at most $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors. Finally, in Sect. 4, we show the tightness of the obtained bound for an infinite family of Cactus graphs.

2 Star Edge Coloring of Unicyclic Cactus Graphs

We first introduce the terminology and notations that we need through the paper. For further information on graph theory concepts and terminology, we refer the reader to Bondy and Murty (2008).

Let G be a graph. We say G is 2-*connected* if between every two vertices there are at least two internally disjoint paths. A *block* in G is a maximal 2-connected subgraph in G. Thus, every block in a Cactus is either an edge or a cycle. A *block graph* H of G is a graph that its vertices are the blocks of G and two vertices of H are adjacent if and only if their corresponding blocks in G intersect in a vertex. Clearly, every block graph is a tree.

For every block C in Cactus G, we consider three types of edge, as follows:

$$E_1(C) = \{e : e \in E(C)\}.$$
$$E_2(C, x) = \{e = xy : x \in V(C), y \notin V(C)\},$$
$$E_2(C) = \bigcup_{x \in V(C)} E_2(C, x).$$
$$E_3(C, e) = \{e' : e' \notin E_1(C) \cup E_2(C), e' \text{ is adjacent to } e, e \in E_2(C)\},$$
$$E_3(C) = \bigcup_{e \in E_2(C)} E_3(C, e).$$

In Fig. 1, a UCC graph with its three types of edges is shown.

In every edge coloring c of G with k colors, we show the color set by $\mathcal{C} := \{1, 2, \ldots, k\}$, the color set of all edges incident to vertex x by $C(x)$, and the color of edge e by $c(e)$. We now present an algorithm to give a star edge coloring of every Δ-semiregular UCC, with $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors.

Theorem 1 *If G is a UCC graph with maximum degree Δ, then*

$$\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1.$$

Proof Without loss of generality, assume that $G = C \cup F$ is Δ-semiregular. In Algorithm 1, we present a star edge coloring of G with $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors.

Algorithm 1 Star edge coloring of Δ-semiregular UCC Graphs

Input: Graph $G = C \cup F$ as a Δ-semiregular UCC, color set $\mathcal{C} = \{1, \ldots, \lfloor \frac{3\Delta}{2} \rfloor + 1\}$.
Output: Star edge coloring of G with color set $\mathcal{C}$.

Give an optimum star edge coloring of C
Set $\mathcal{C}' = \mathcal{C} \setminus \cup_{e \in E_1(C)} c(e)$.
Color edges of $E_2(C)$ with $\Delta - 2$ colors from $\mathcal{C}'$.
for each x in $V(C)$ **do**
  label incident edges to x such that e_i is the i-th edge incident to x in T_x in clockwise order.
  for i from 1 to $\Delta - 2$ **do**
    Color $\lfloor \frac{\Delta}{2} \rfloor + 1$ edges of $E_3(C, e_i)$ with colors of $\mathcal{C} \setminus C(x)$.
    if *i is even* **then**
      Use colors of edges $e_{2k+1}, e_{2k'}$, where $2k + 1 < i$ and $i < 2k'$, for uncolored edges in $E_3(C, e_i)$.
    else
      Use colors of edges $e_{2k}, e_{2k'+1}$, where $2k < i$ and $i < 2k' + 1$, for uncolored edges in $E_3(C, e_i)$.
    end if
  end for
end for

The performance of Algorithm 1 is as follows. We first give an optimum star edge coloring for block C in $G = C \cup F$, where $\chi'_s(C) \leq 4$ [see the proof of Theorem 5.1 in Dvořák et al. (2013)]. Let $\mathcal{C}'$ be the set of colors that are

(a) $E_1(C_i) \cup E_2(C_i) \cup E_3(C_i)$ **(b)** G_{C_i}

Fig. 2 Construction of UCC graph G_{C_i}

not used for coloring C. We choose $\Delta - 2$ colors from $\mathcal{C}'$ for coloring the edges of $E_2(C)$. Assume that $x \in V(C)$ and $e_i \in T_x$ is the ith edge incident to x, where $1 \leq i \leq \Delta - 2$.

We color $\lfloor \frac{\Delta}{2} \rfloor + 1$ edges of $E_3(C, e_i)$ with colors that are not used in $C(x)$. To complete coloring of the edges in $E_3(C, e_i)$, we have two possibilities: index i is even or odd. If i is even (or odd), we use colors of incident edges to x with even indices more (or less) than i and odd indices less (or more) than i. Note that in both cases, there exist at least $\lfloor \frac{\Delta}{2} \rfloor - 1$ colors for the uncolored edges in $E_3(C, e_i)$. Thus, we can color all edges in G with at most $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors.

We now prove that this coloring is a star edge coloring. For this purpose, we first check the coloring of paths of length 4 in T_x. Consider two arbitrary edges $e_j = xy_j$ and $e_k = xy_k$ in $E_2(C, x)$. Without loss of generality, assume that $j < k$. If the parity of j and k is different, then $c(e_k) \notin C(y_j)$. Similarly, if the parity of j and k is the same, then $c(e_j) \notin C(y_k)$. Thus, we have no bicolored path of length 4 in T_x. Moreover, since the color set of the edges in $E_3(C)$ is disjoint from the color set of C, the obtained coloring is a star edge coloring of G. □

3 Star Edge Coloring of Cactus Graphs

In this section, we prove Conjecture 1 for Cactus graphs.

Theorem 2 *If G is a Cactus with maximum degree* Δ, *then*

$$\chi_s'(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1.$$

Proof In Bezegová et al. (2016), it is proved that the star chromatic index of outerplanar graphs with maximum degree 3 is at most 5. Thus, we prove the statement for $\Delta \geq 4$.

Let $\sigma = (C_1, \ldots, C_t)$ be an enumeration of blocks in G, in the order in which they are visited by *breadth first search* (BFS) in block graph G. The BFS is a traversing algorithm where we start traversing from a selected vertex and explore all of the neighbor vertices at the present level prior to moving on to the vertices at the next level.

For every block C_i, $1 \leq i \leq t$, we construct a UCC graph G_{C_i}, corresponding to the three types of edges of C_i in G, as follows. Let D be a cycle of length 3 or 4 in $E_2(C_i) \cup E_3(C_i)$ with two incident edges $e_1 = xy_1$ and $e_2 = xy_2$ to vertex $x \in C_i$. We remove edges of $E(D) \cap E_3(C_i)$ and add two new vertices y_1' and y_2' connecting to y_1 and y_2, respectively. We apply this process for every cycle in $E_2(C_i) \cup E_3(C_i)$. The final graph is UCC graph G_{C_i} (Fig. 2).

We now present the following algorithm, to provide a star edge coloring for every Δ-semiregular Cactus with $\lfloor \frac{3\Delta}{2} \rfloor + 1$ colors.

Algorithm 2 Star edge coloring of Cactus graphs

Input: Δ-semiregular Cactus G, enumeration $\sigma = (C_1, \ldots, C_t)$ for the blocks of G obtained by BFS in block graph G, color set $\mathcal{C} = \{1, \ldots, \lfloor \frac{3\Delta}{2} \rfloor + 1\}$.

Output: Star edge coloring of G with color set $\mathcal{C}$.

1: **for** i from 1 to t **do**
2: Construct UCC graph G_{C_i} from the three types edges of C_i in G.
3: **if** $i = 1$ **then**
4: Apply Algorithm 1 for G_{C_1}.
5: **else**
6: Consider restriction partial coloring of G for their corresponding edges in G_{C_i}.
7: **for** Every vertex x of C_i **do**
8: **if** Edges of T_x is uncolored **then**
9: Run lines 2,3, and 5 to 13 in Algorithm 1 for $C_i \cup T_x$.
10: **else if** Eedges of $E_3(C_i) \cap T_x$ are uncolored **then**
11: Run lines 5 to 13 in Algorithm 1 for $C_i \cup T_x$.
12: **end if**
13: **end for**
14: **end if**
15: Set c_i as the obtained coloring of G_{C_i}.
16: Color uncolored edges of $E_1(C_i) \cup E_2(C_i)$ in G in G in the same fashion as the corresponding edges in G_{C_i}.
17: **for** every uncolored cycle $D_x \neq C_i$ with vertex x in C_i **do**
18: Set e_1 and e_2 as the edges incident to x in D_x, where $E_3(C_i, e_1)$ is colored before $E_3(C_i, e_2)$ in G_{C_i}.
19: Enumerate edges of D_x as $D_x = e_1, e_2, \ldots, e_n$, where n is the length of D_x.
20: Set $C'(x) = \mathcal{C} \setminus C(x)$.
21: **if** $n = 0 \pmod 3$ **then**
22: Select edges $e \in E_3(C_i, e_1)$ and $e' \in E_3(C_i, e_2)$ of G_{C_i} with the same color c' in $C'(x)$.
23: Complete the coloring of D_x in G by coloring pattern $\underbrace{c_i(e_1), c_i(e_2), c'}, \ldots$.
24: **else if** $n = 1 \pmod 3$ **then**
25: Select edge $e' \in E_3(C_i, e_2)$ in G_{C_i}, with $c_i(e') \in C'(x)$.
26: **if** $n = 4$ and $\Delta \neq 4$ **then**
27: Select edge $e \in E_3(C_i, e_1)$, with $c_i(e) \in C(x)$.
28: Complete the coloring of D_x using pattern $c_i(e_1), c_i(e_2), c_i(e'), c_i(e)$.
29: **else**
30: Complete the coloring of D_x using coloring pattern $c_i(e_1), \underbrace{c_i(e_2), c_i(e_1), c_i(e')}, \ldots$.
31: **end if**
32: **else**
33: Select edges $e \in E_3(C_i, e_1)$ and $e' \in E_3(C_i, e_2)$ in G_{C_i}, with $c_i(e) \in C'(x)$ and $c_i(e') = c_i(e_1)$.
34: **if** $n = 5$ **then**
35: Choose color $c' \in C'(x) \setminus c_i(e_1)$.
36: Complete the coloring of D_x using coloring pattern $c_i(e_1), c_i(e_2), c_i(e_1), c', c_i(e)$.
37: **else**
38: Complete the coloring of D_x using coloring pattern $c_i(e_1), c_i(e_2), \underbrace{c_i(e_1), c_i(e), c_i(e_2)}, \ldots, \underbrace{c_i(e_1), c_i(e), c_i(e_2)}, c_i(e_1), c_i(e_2), c_i(e)$.
39: **end if**
40: **end if**
41: **end for**
42: Color the remaining uncolored edges of $E_3(C_i)$ in G in the same fashion as the corresponding edges in G_{C_i}.
43: **end for**

Algorithm 2 runs as follows. In the ith round, we consider the ith block of G in enumeration σ and construct graph G_{C_i}. Note that for $i > 1$, G_{C_i} has a partial star edge coloring that edges of C_i and some edges of $E_2(C_i) \cup E_3(C_i)$ are colored. The edges in a common level of T_x are either already colored or not. In lines 3 to 14 of Algorithm 2, we complete the coloring of G_{C_i}.

We now color edges of $E_1(C_i) \cup E_2(C_i)$ in G in the same fashion as the corresponding edges in G_{C_i}. Let H_{C_i} be the induced subgraph of G on $E_1(C_i) \cup E_2(C_i) \cup E_3(C_i)$ and all cycles share a vertex with C_i. We obtain a coloring of H_{C_i}, as follows.

We first complete coloring of every uncolored cycle that shares a vertex with C_i in G, according to its length (see lines 17 to 41 in the algorithm). For example, let $D_x = e_1, e_2, \ldots, e_n$ be an uncolored cycle with two edges e_1 and e_2 incident to $x \in C_i$. For D_x with different lengths, we demonstrate its coloring in Figs. 3 and 4, by the following assumptions. Let $C(x) = \{1, \ldots, \Delta\}$, $c(e_1) = 1$, $c(e_2) = 2$, $\{A, B\} \subseteq C'(x)$, $A \neq B$, $\lambda \in (C(x_1) \cap C(x)) \setminus \{1, 2\}$, and the dashed edges are in $G_{C_i} \setminus G$.

Finally, we color the remaining edges of $E_3(C_i)$ in H_{C_i} in the same fashion as the corresponding edges in G_{C_i}. Note that, in lines 3 to 14, edges of $E_3(C_i, e_1)$ are colored before $E_3(C_i, e_2)$. Thus, $c_i(e_2)$ is not used for the edges in $E_3(C_i, e_1)$ and all colors of $C'(x)$ are used in $E_3(C_i, e_1)$ and $E_3(C_i, e_2)$. Hence, it is easy to check that the coloring of $C_i \cup T_x \cup D$ is a star edge coloring.

We now, by induction on t, prove that the obtained coloring is a star edge coloring of G. For $i = 1$, by applying Algorithm 1 in line 4 of Algorithm 2, the statement is obvious. Now, assume that after the $(i-1)$th round in

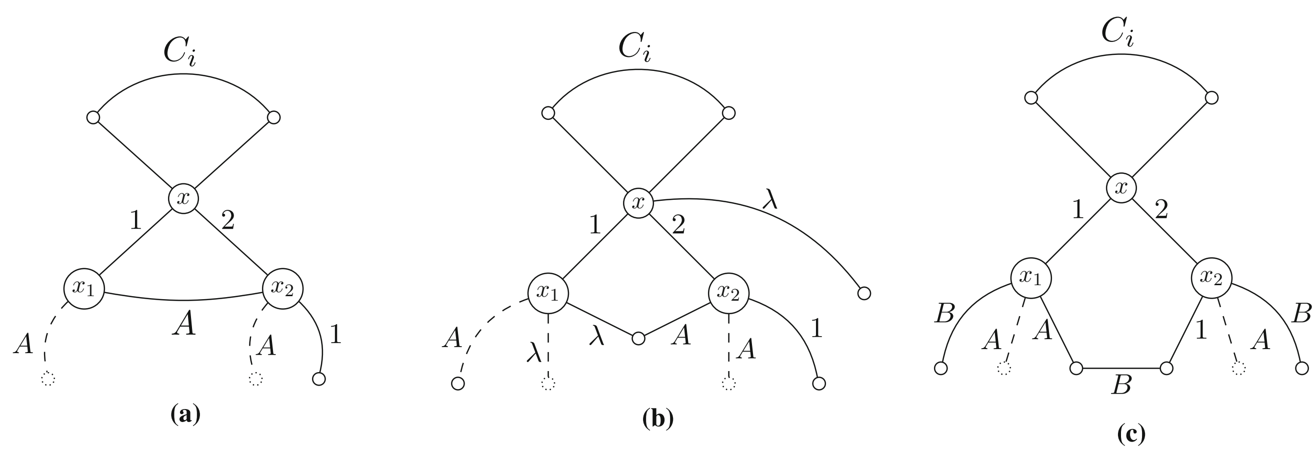

Fig. 3 Star edge coloring of cycle D_x of length 3, 4, and 5

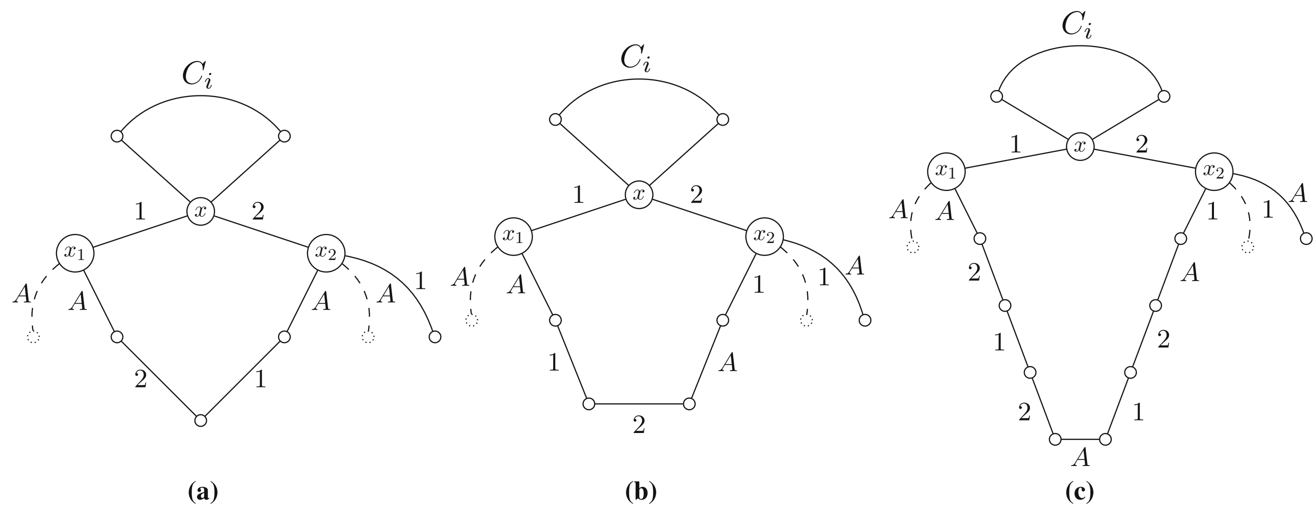

Fig. 4 Star edge coloring of cycle D_x of length at least 6

Algorithm 2 there is no bicolored path of length 4, but bicolored path $P := e_1, e_2, e_3, e_4$ appears after the ith round. We have two possibilities: All edges of P belong to H_{C_i} or not. In the first case, since for each vertex x of C_i there is no bicolored path in $C_i \cup T_x \cup D_x$, two adjacent edges of P belong to C_i or are in the same T_x. Therefore, e_2 or e_3 is in C_i and is colored before the ith round. Without loss of generality, assume that $e_3 \in C_i$. If e_1 or e_2 is uncolored before the ith round, then for some x in C_i, $e_1 \in T_x$ and by coloring T_x in the ith round, colors of e_1 and e_3 are different, that is a contradiction. Thus, edges e_1 and e_2 are also colored before the ith round.

Now, let P has some edges out of H_{C_i}. In this case, because of enumeration σ, P has three colored edges before the ith round. According to this argument, it follows that $e_1e_2e_3$ is a bicolored path obtained before the ith round. If $e_2 \notin C_i$, then all paths of length 4 that contain P are colored before the ith round and by the induction hypothesis, P is not bicolored. Thus, $e_2 \in C_i$ and is colored before the ith round. Since $e_4 \in E_2(C_i) \cup E_3(C_i)$ and is colored in the ith round by Algorithm 1, color of e_4 is not the same as color of e_2. Therefore, P is not bicolored, that again is a contradiction. Hence, the obtained coloring in Algorithm 2 is a star edge coloring of G. □

Remark Note that the running time of Algorithms 1 and 2 are polynomial. Thus, for every Cactus graph with maximum degree Δ, the star edge coloring with $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$ colors, given in Theorem 2, is obtained in polynomial time.

4 Tightness of the Bound

In this section, we prove that the given bound in Theorem 2 is tight and there is an infinite family of Cactus graphs that achieve the bound $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$. First, we need to see some facts about the star edge coloring of trees.

Lemma 1 *Let Δ be an odd positive integer and T_v be a Δ-semiregular rooted tree of height two. In every star edge coloring c of T_v with color set $\mathcal{C} = \{1, \ldots, \lfloor \frac{3\Delta}{2} \rfloor\}$ and for every two neighbors x and y of the root v, we have*

(a) $|C(x) \cap C(y)| \geq \frac{\Delta+1}{2}$, $C'(v) = \mathcal{C} \backslash C(v) \subset C(x)$.

(b) Every color of $C(v)$ is used for $\frac{\Delta-1}{2}$ non-incident edges to v.

(c) $c(vx) \in C(y)$ or $c(vy) \in C(x)$.

Proof It is clear that

$$|C(x) \cap C(y)| \geq 2\Delta - \left\lfloor \frac{3\Delta}{2} \right\rfloor = \frac{\Delta+1}{2}.$$

Let n_i be the numbers of edges colored with i in $T_v \backslash \{v\}$. Obviously, $C(x) \cap C'(v) \leq \frac{\Delta-1}{2}$. Moreover, for every two neighbors x and y of v, if color $c(vx)$ is used for some incident edges to y, then color $c(vy)$ is not used for the incident edges to x, and vice versa. Thus, for each color $i \in C(v)$, we have $n_i \leq \frac{\Delta-1}{2}$. Also, each color of $C'(v)$ can be used for the incident edges of each neighbor of v. Since T_v has $\Delta(\Delta-1)$ leaves, we must have

$$\Delta(\Delta-1) = \sum_{i \in C(v)} n_i + \sum_{i \in C'(v)} n_i \leq \sum_{i \in C(v)} \frac{\Delta-1}{2} + \sum_{i \in C'(v)} \Delta = \Delta(\Delta-1),$$

which implies that if $i \in C'(v)$, then $n_i = \Delta$ which proves (a). Moreover, if $i \in C(v)$, then $n_i = \frac{\Delta-1}{2}$, that proves (b).

To prove (c), by contrary, suppose that $c(vx) \notin C(y)$ and $c(vy) \notin C(x)$. Then, colors $c(vx)$ and $c(vy)$ can be used for at most $(\Delta-2) - \frac{\Delta-1}{2} < \frac{\Delta-1}{2}$ non-incident edges to v, that contradicts (b). □

Theorem 3 *For every odd positive integer $\Delta \geq 3$, there exists a Cactus G with maximum degree Δ, in which $\chi'_s(G) = \lfloor \frac{3\Delta}{2} \rfloor + 1$.*

Proof For every positive odd integer Δ, we construct a Cactus that achieves the bound. For this purpose, let T_v be a Δ-semiregular tree of height three with root v, and x and y are two neighbors of v. For every vertex $u \neq v$ in T_v, we denote the neighbors of u by $\{u_0, u_1, \ldots, u_{\Delta-1}\}$, where u_0 is the parent of u.

We construct Cactus G_T from T_v by adding edge xy and removing vertices $x' = x_{\Delta-1}$ and $y' = y_{\Delta-1}$. Obviously, if G_T admits a star edge coloring ϕ with $\lfloor \frac{3\Delta}{2} \rfloor$ colors, then T_v has star edge coloring c with $\lfloor \frac{3\Delta}{2} \rfloor$ colors, where

$$c(e) = \begin{cases} \phi(e) & \text{if} e \in G_T, \\ \phi(xy) & \text{if} e = xx' \text{ or } e = yy', \\ \phi(xx_{i-1}) & \text{if} e = y'y'_i, \quad 1 \leq i \leq \Delta - 1, \\ \phi(yy_{i-1}) & \text{if} e = x'x'_i, \quad 1 \leq i \leq \Delta - 1. \end{cases}$$

By Lemma 1(c), $\phi(vx) \notin C(y)$ or $\phi(vy) \notin C(x)$, and hence $\phi(xy) \notin C(v)$. Moreover, without loss of generality, assume that $\phi(vy) \in C(x)$. Lemma 1(a) implies that $C(x) \cap C(y) = \frac{\Delta+1}{2}$. Therefore, color $\phi(xy)$ can be used for at most $(\Delta-1) - \frac{\Delta+1}{2} < \frac{\Delta-1}{2}$ incident edges to the neighbors of x, except v. That contradicts Lemma 1(b) for Δ-semiregular subtree T_x in T_v. Therefore, $\chi'_s(G_T) \geq \lfloor \frac{3\Delta}{2} \rfloor + 1$. Hence, by Theorem 2 we have $\chi'_s(G_T) = \lfloor \frac{3\Delta}{2} \rfloor + 1$. □

Theorem 4 *There exists Cactus G with maximum degree 6, where $\chi'_s(G) = 10$*

Proof Let G be the Cactus shown in Fig. 5. We show that G has no star edge coloring with color set $\mathcal{C} = \{1, \ldots, 9\}$. Assume that $C := x_1x_2x_3$ is the cycle of length 3 in G colored by 1, 2, and 3. In every star edge coloring $E_1(C) \cup E_2(C)$ with colors $\mathcal{C}$, there is a vertex x_i that the color set of edges in $E_2(C, x_i)$ is a subset of $\{4, 5, 6, 7, 8\}$ and the colors of at least two edges in $E_2(C, x_i)$ are used at least two times in the color set of edges in $E_2(C)$. Without loss of generality, assume that $i = 3$ and $C(x_3) = \{1, 3, 4, 5, 6, 7\}$. Let $D := x_3y_1y_2y_3$ and $c(x_3y_1) = 4$, $c(x_3y_3) = 5$. The common possible colors that we can use for edges incident to y_1 and y_3 are $\{2, 8, 9\} = \mathcal{C} \backslash C(x_3)$, we color three edges in $E_2(D, y_1)$ and $E_2(D, y_3)$ with these colors. On the other hand, colors 3 and 1 are usable for coloring an edge in $E_2(D, y_1)$ and $E_2(D, y_3)$, respectively. If we set $c(y_1y_2) = A \in \{2, 8, 9\}$ and $c(y_2y_3) = B \in \{2, 8, 9\}$ or we set $c(y_1y_2) = A \in \{2, 8, 9\}$ and $c(y_2y_3) = 4$, then we have a bicolored path of length 4 in G. Otherwise, we must have

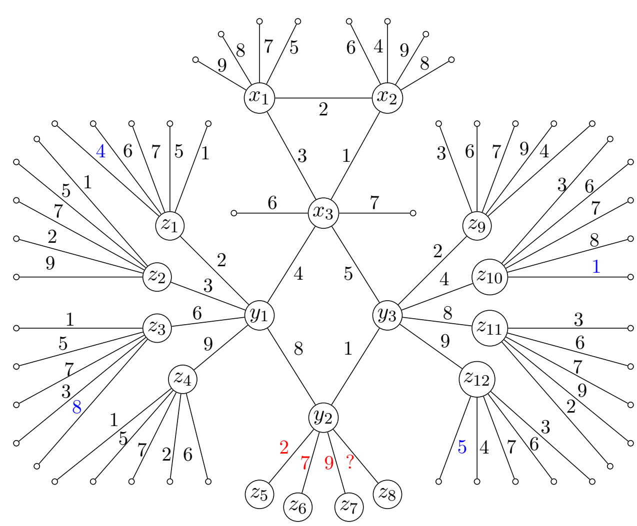

Fig. 5 A Cactus G with $\Delta = 6$ and $\chi_s(G) = 10$

$c(y_1y_2) = A \in \{2, 8, 9\}$ and $c(y_2y_3) = 1$ (or $c(y_1y_2) = 3$ and $c(y_2y_3) = A$).

Now, we show that this is impossible as well. For this purpose, let $A = 8$ and color the edges in $E_3(D, y_1z_i)$, $1 \leq i \leq 4$, with colors $\{1, \ldots, 9\} \backslash \{8\}$ such that there is no bicolored path of length 4. First, color one edge in $E_2(D, y_2)$ with color 7. According to colors of $E_3(D, y_1z_1)$, we can color two edges of $E_2(D, y_2)$ with 2 and 9. On the other hand, the only possible edge that can be colored with color 1 is an edge in $E_3(D, y_3z_{12})$. This implies that there is no possible color to color edge y_2z_8 with a color in the color set of edges in $E_2(D, y_3) \cup E_3(D, y_3z_9) \cup E_3$ (D, y_3z_10). By a similar discussion, it can be shown that if we set $c(y_1y_2) = 3$ and $c(y_2y_3) = 1$ or $c(y_1y_2) = 6$ and $c(y_2y_3) = 1$ or $c(y_1y_2) = 6$ and $c(y_2y_3) = A$, then coloring of G with 9 colors is impossible. $\square$

Conjecture 2 *For every even integer* $\Delta \geq 6$, *there exists a* Δ-*semiregular Cactus* G *with star chromatic index* $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$.

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