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The metric dimension of the lexicographic product of graphs

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ABSTRACT

For a set W of vertices and a vertex v in a connected graph G, the k-vector $r_W(v) = (d(v, w_1), \ldots, d(v, w_k))$ is the *metric representation* of v with respect to W, where $W = \{w_1, \ldots, w_k\}$ and $d(x, y)$ is the distance between the vertices x and y. The set W is a *resolving set* for G if distinct vertices of G have distinct metric representations with respect to W. The minimum cardinality of a resolving set for G is its *metric dimension*. In this paper, we study the metric dimension of the lexicographic product of graphs G and H, denoted by $G[H]$. First, we introduce a new parameter, the *adjacency dimension*, of a graph. Then we obtain the metric dimension of $G[H]$ in terms of the order of G and the adjacency dimension of H.

1. Introduction

In this section, we present some definitions and known results that are necessary to prove our main theorems. Throughout this paper, G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use $\overline{G}$ for the complement of G. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path joining u and v in G. Also, $N_G(v)$ is the set of all neighbors of vertex v in G. We write these simply as $d(u, v)$ and $N(v)$ when no confusion can arise. The adjacency and non-adjacency relations are denoted by $\sim$ and $\not\sim$, respectively. We use P_n and C_n to denote the isomorphism classes of n-vertex paths and cycles, respectively. We use $\langle v_1, \ldots, v_n\rangle$ and $[v_1, \ldots, v_n]$ to denote specific n-vertex paths and cycles with vertices $v_1, \ldots, v_n$ in order. We also use notation $\mathbf{1}$ for the vector $(1, \ldots, 1)$ and $\mathbf{2}$ for the vector $(2, \ldots, 2)$.

For $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r_W(v) = (d(v, w_1), \ldots, d(v, w_k))$$

is the *metric representation* of v with respect to W. The set W is a *resolving set* for G if the vertices of G have distinct metric representations. In this case, we say that W *resolves* G. Elements in a resolving set are *landmarks*. A resolving set W for G with minimum cardinality is a *metric basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\mu(G)$. The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [15] and by Harary and Melter [11]. For more results related to these concepts see [2,3,7,9,17].

We say that a set W *resolves* a set T of vertices in G if the metric representations of vertices in T with respect to W are distinct. When $W = \{x\}$, we say that the vertex x resolves T. To determine whether a given set W is a resolving set for G, it is sufficient to look at the metric representations of vertices in $V(G) \setminus W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$.

Two distinct vertices u and v are *twins* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. We write $u \equiv v$ if and only if $u = v$ or u and v are twins. In [12], it is proved that "$\equiv$" is an equivalence relation. The equivalence class of vertex v is denoted by v^*. Hernando

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et al. [12] proved that v^* is a clique or an independent set in G. As in [12], we say that v^* is of type 1, K, or N if v^* is a class of size 1, a clique of size at least 2, or an independent set of size at least 2, respectively. We denote the number of equivalence classes of G with respect to "$\equiv$" by $\iota(G)$. We denote by $\iota_K(G)$ and $\iota_N(G)$ the number of classes of type K and type N in G, respectively. We also use $a(G)$ and $b(G)$ for the number of vertices in G belonging to classes of type K or type N, respectively. Clearly, $\iota(G) = n(G) - a(G) - b(G) + \iota_N(G) + \iota_K(G)$.

Observation 1.1 *([12]). If u and v are twins in a graph G, and W resolves G, then u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves G.*

Theorem 1.2 *([8]). If G is a connected graph of order n, then*

(i) $\mu(G) = 1$ *if and only if* $G = P_n$ *and*
(ii) $\mu(G) = n - 1$ *if and only if* $G = K_n$.

Let G and H be two graphs with disjoint vertex sets. The *join* of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.

Theorem 1.3 *([4,5]).*

(i) *If* $n \notin \{3, 6\}$, *then* $\mu(C_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$,
(ii) *If* $n \notin \{1, 2, 3, 6\}$, *then* $\mu(P_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

The *Cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H) = \{(v, u): v \in V(G), u \in V(H)\}$, where two vertices (v, u) and (v', u') are adjacent if $u = u'$ and $vv' \in E(G)$, or $v = v'$ and $uu' \in E(H)$. The metric dimension of the Cartesian product of graphs is studied by Caceres et al. in [6]. They obtained the metric dimension of $G \square H$ when $G, H \in \{P_n, C_n, K_n\}$.

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (v, u) and (v', u') are adjacent if $vv' \in E(G)$, or $v = v'$ and $uu' \in E(H)$. When the order of G is at least 2, it is easy to see that $G[H]$ is connected if and only if G is connected. For more information about the lexicographic product of graphs, see [13].

This paper studies the metric dimension of the lexicographic product of graphs. The main goal of Section 2 is introducing a new parameter called the "adjacency dimension". In Section 3, we determine the metric dimension of some lexicographic products of the form $G[H]$ in terms of the order of G and the adjacency dimension of H. In Corollaries 3.12 and 3.13, we use Theorems 3.3, 3.5, 3.7 and 3.9 to obtain the exact value of the metric dimension of $G[H]$, where $G = C_n$ for $n \geq 5$ or $G = P_n$ for $n \geq 4$, and $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_{m_1,\ldots,m_t}, \overline{K}_{m_1,\ldots,m_t}\}$.

2. Adjacency resolving sets

Khuller et al. [14] considered the application of the metric dimension of a connected graph in robot navigation. In that sense, a robot moves from node to node of a graph. If the robot knows its distances to a sufficiently large set of landmarks, then its position on the graph is uniquely determined. This suggests the problem of finding the minimum number of landmarks needed, and where they should be located, so that the distances to the landmarks uniquely determine the robot's position on the graph. The solutions of these problems are the *metric dimension* and a *metric basis* of the graph, respectively.

Now let there exist a large number of landmarks, but suppose that the cost of computing distance is too much for the robot. In this case, we want the robot to be able to determine its position only from landmarks adjacent to it. Now the problem is that of finding the minimum number of landmarks needed, and where they should be located, so that adjacency and non-adjacency to the landmarks uniquely determine the robot's position on the graph. This problem motivates introducing *adjacency resolving sets* in graphs.

Definition 2.1. Let G be a graph, and let $W = \{w_1, \ldots, w_k\} \subseteq V(G)$. For each vertex $v \in V(G)$, the *adjacency representation* of v with respect to W is the k-vector

$$\hat{r}_W(v) = (a_G(v, w_1), \ldots, a_G(v, w_k)),$$

where

$$a_G(v, w_i) = \begin{cases} 0 & \text{if } v = w_i, \\ 1 & \text{if } v \sim w_i, \\ 2 & \text{if } v \nsim w_i. \end{cases}$$

The set W is an *adjacency resolving set* for G if the vectors $\hat{r}_W(v)$ for $v \in V(G)$ are distinct. The minimum cardinality of an adjacency resolving set is the *adjacency dimension* of G, denoted by $\hat{\mu}(G)$. An adjacency resolving set of cardinality $\hat{\mu}(G)$ is an *adjacency basis* of G.

Babai [1] studied the idea of adjacency resolving sets for strongly regular graphs, in the context of the graph isomorphism problem, but he called them distinguishing sets. Also, Slater et al. [10,16] introduced some other related parameters.

By the definition, if G is a connected graph with diameter 2, then $\hat{\mu}(G) = \mu(G)$. The converse is false; it can be seen that $\hat{\mu}(C_6) = 2 = \mu(C_6)$ while $\mathrm{diam}(C_6) = 3$.

In the following, we obtain some useful results on the adjacency dimension of graphs.

Proposition 2.2. *If a graph G is connected, then $\hat{\mu}(G) \geq \mu(G)$.*

Proof. Let W be an adjacency basis of G. For distinct vertices $u, v \in V(G)$, there exists a vertex $w \in W$ such that $a_G(u, w) \neq a_G(v, w)$. Therefore $d_G(u, w) \neq d_G(v, w)$, and hence W is a resolving set for G. Thus every adjacency resolving set for G is a resolving set and $\mu(G) \leq \hat{\mu}(G)$. $\square$

The next proposition follows from the fact that there exists a bijection between adjacency representations in G and $\overline{G}$, by interchanging the values 1 and 2.

Proposition 2.3. *For every graph G, $\hat{\mu}(G) = \hat{\mu}(\overline{G})$.*

Let G be a graph of order n. It is easy to see that $1 \leq \hat{\mu}(G) \leq n - 1$. In the following proposition, we characterize all graphs G with $\hat{\mu}(G) = 1$ and all graphs G of order n with $\hat{\mu}(G) = n - 1$.

Proposition 2.4. *If G is a graph of order n, then*

(i) $\hat{\mu}(G) = 1$ *if and only if* $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$.
(ii) $\hat{\mu}(G) = n - 1$ *if and only if* $G = K_n$ *or* $G = \overline{K}_n$.

Proof. (i) It is easy to see that $\hat{\mu}(G) = 1$ for $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$. Conversely, let G be a graph with $\hat{\mu}(G) = 1$. If G is connected, then by Proposition 2.2, $\mu(G) \leq \hat{\mu}(G) = 1$. Thus by Theorem 1.2, $G = P_n$. If $n \geq 4$, then each vertex of P_n has at least two neighbors or two non-neighbors in P_n; consequently 1-subsets of $V(P_n)$ are not adjacency resolving sets for P_n, and hence $\hat{\mu}(P_n) \geq 2$. Therefore $n \leq 3$. If G is disconnected and $\hat{\mu}(G) = 1$, then $\overline{G}$ is connected and by Proposition 2.3, $\hat{\mu}(\overline{G}) = 1$. Thus $\overline{G} = P_n$, $n \in \{2, 3\}$. Consequently $G = \overline{P}_2$ or $G = \overline{P}_3$.

(ii) By Proposition 2.2, we have $n - 1 = \mu(K_n) \leq \hat{\mu}(K_n)$. On the other hand, $\hat{\mu}(G) \leq n - 1$. Therefore $\hat{\mu}(K_n) = n - 1$ and by Proposition 2.3, $\hat{\mu}(\overline{K}_n) = \hat{\mu}(K_n) = n - 1$. Conversely, let G be connected with $\hat{\mu}(G) = n - 1$. Suppose on the contrary that $G \neq K_n$. Thus P_3 is an induced subgraph of G. Let $P_3 = (x_1, x_2, x_3)$. Therefore $a_G(x_2, x_1) = 1$ and $a_G(x_3, x_1) = 2$. Consequently $V(G) \setminus \{x_2, x_3\}$ is an adjacency resolving set for G of cardinality $n - 2$. That is, $\hat{\mu}(G) \leq n - 2$, which is a contradiction. Hence $G = K_n$. If G is disconnected with $\hat{\mu}(G) = n - 1$, then $\overline{G}$ is connected and by Proposition 2.3, $\hat{\mu}(\overline{G}) = n - 1$. Thus $\overline{G} = K_n$. $\square$

Lemma 2.5. *If u is a vertex of degree $n(G) - 1$ in a connected graph G, then G has a metric basis that does not include u.*

Proof. Let B be a metric basis of G that contains u. Thus $r_{B\setminus\{u\}}(u) = \mathbf{1}$. Since B is a metric basis of G, there exist two vertices $v, w \in V(G) \setminus (B \setminus \{u\})$ such that $r_{B\setminus\{u\}}(v) = r_{B\setminus\{u\}}(w)$ and $d_G(u, v) \neq d_G(u, w)$. If $u \notin \{v, w\}$, then $d(u, v) = d(u, w) = 1$, which is a contradiction. Hence $u \in \{v, w\}$, say $u = v$. Therefore $r_{B\setminus\{u\}}(w) = r_{B\setminus\{u\}}(u) = \mathbf{1}$ and for each $x, y \in V(G) \setminus \{u, w\}$, $r_{B\setminus\{u\}}(x) \neq r_{B\setminus\{u\}}(y)$. Note that $r_B(w) = \mathbf{1}$, because $u \sim w$. Since B is a metric basis of G, w is the unique vertex of G whose metric representation with respect to B is $\mathbf{1}$. It implies that w is the unique vertex of $V(G) \setminus B$ with $r_{B\setminus\{u\}}(w) = \mathbf{1}$. Therefore the set $(B \setminus \{u\}) \cup \{w\}$ is a metric basis of G that does not contain u. $\square$

Proposition 2.6. *For every graph G, $\mu(G \vee K_1) - 1 \leq \hat{\mu}(G) \leq \mu(G \vee K_1)$. Moreover, $\hat{\mu}(G) = \mu(G \vee K_1)$ if and only if G has an adjacency basis with respect to which no vertex has adjacency representation $\mathbf{1}$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(K_1) = \{u\}$. Note that $d_{G\vee K_1}(v_i, v_j) = a_G(v_i, v_j)$ for $1 \leq i, j \leq n$. By Lemma 2.5, $G \vee K_1$ has a metric basis $B = \{b_1, \ldots, b_k\}$ such that $u \notin B$. Therefore

$$r_B(v_i) = (d_{G\vee K_1}(v_i, b_1), \ldots, d_{G\vee K_1}(v_i, b_k)) = \hat{r}_B(v_i)$$

for each v_i. Thus B is an adjacency resolving set for G, and $\hat{\mu}(G) \leq \mu(G \vee K_1)$.

Now let W be an adjacency basis of G. Since $d_{G\vee K_1}(v_i, w) = a_G(v_i, w)$ for $1 \leq i \leq n$ and $w \in W$, we have $r_W(v_i) = \hat{r}_W(v_i)$ for every i. Hence W resolves $V(G \vee K_1) \setminus \{u\}$, and $\mu(G \vee K_1) - 1 \leq \hat{\mu}(G)$. On the other hand, $r_W(u) = \mathbf{1}$. Therefore W is a resolving set for $G \vee K_1$ if and only if $\hat{r}_W(v_i) \neq \mathbf{1}$ for every v_i. Since $\hat{\mu}(G) \leq \mu(G \vee K_1)$, we have $\hat{\mu}(G) = \mu(G \vee K_1)$ if and only if $\hat{r}_W(v_i) \neq \mathbf{1}$ for every v_i. $\square$

Proposition 2.7. *If $n \geq 4$, then $\hat{\mu}(C_n) = \hat{\mu}(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.*

Proof. If $n \leq 8$, then case analysis yields $\hat{\mu}(C_n) = \hat{\mu}(P_n) = \lfloor \frac{2n+2}{5} \rfloor$. Now let $G \in \{P_n, C_n\}$, and $n \geq 9$. By Theorem 1.3, $\mu(G \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor \geq 4$. Hence by Proposition 2.6, we have $\hat{\mu}(G) \geq 3$. If W is an adjacency basis of G, then for each vertex $v \in V(G)$, $\hat{r}_W(v) \neq \mathbf{1}$, because v has at most two neighbors. Therefore by Proposition 2.6, $\hat{\mu}(G) = \hat{\mu}(G \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$. $\square$

Proposition 2.8. *If $K_{m_1,\ldots,m_t}$ is the complete t-partite graph with r parts of size at least 2 and the other parts of size 1 and $\sum_{i=1}^{t} m_i = m$, then*

$$\hat{\mu}(K_{m_1,\ldots,m_t}) = \mu(K_{m_1,\ldots,m_t}) = \begin{cases} m-r-1 & \text{if } r \neq t, \\ m-r & \text{if } r = t. \end{cases}$$

Proof. Since $\operatorname{diam}(K_{m_1,\ldots,m_t}) = 2$, we have $\hat{\mu}(K_{m_1,\ldots,m_t}) = \mu(K_{m_1,\ldots,m_t})$. Let M_i be the partite set of size m_i. For $1 \le i \le r$, all vertices of M_i are non-adjacent twins. Also, all vertices of $\bigcup_{i=r+1}^{t} M_i$ are adjacent twins. Let x_i be a fixed vertex in M_i for $1 \le i \le r$. If $r = t$, then by Observation 1.1, $\mu(K_{m_1,\ldots,m_t}) \ge \sum_{i=1}^{t} m_i - r$. Also, the set $\bigcup_{i=1}^{t} M_i \setminus \{x_1, \ldots, x_r\}$ is a resolving set for $K_{m_1,\ldots,m_t}$ with cardinality $\sum_{i=1}^{t} m_i - r$. Thus $\mu(K_{m_1,\ldots,m_t}) = \sum_{i=1}^{t} m_i - r = m - r$. If $r \neq t$, then $\bigcup_{i=r+1}^{t} M_i \neq \emptyset$. Let $x_{r+1} \in \bigcup_{i=r+1}^{t} M_i$. Observation 1.1 implies that $\mu(K_{m_1,\ldots,m_t}) \ge \sum_{i=1}^{t} m_i - r - 1$. On the other hand, the set $\bigcup_{i=1}^{t} M_i \setminus \{x_1, \ldots, x_{r+1}\}$ is a resolving set for $K_{m_1,\ldots,m_t}$ with cardinality $\sum_{i=1}^{t} m_i - r - 1 = m - r - 1$. $\square$

3. The lexicographic product of graphs

Throughout this section, G is a connected graph of order n with $V(G) = \{v_1, \ldots, v_n\}$, and H is a graph of order m with $V(H) = \{u_1, \ldots, u_m\}$. Therefore $G[H]$ is a connected graph. For convenience, we denote the vertex (v_i, u_j) of $G[H]$ by v_{ij}. Note that for distinct vertices $v_{ij}, v_{rs} \in V(G[H])$,

$$d_{G[H]}(v_{ij}, v_{rs}) = \begin{cases} d_G(v_i, v_r) & \text{if } v_i \neq v_r, \\ 1 & \text{if } v_i = v_r \text{ and } u_j \sim u_s, \\ 2 & \text{if } v_i = v_r \text{ and } u_j \nsim u_s. \end{cases}$$

In other words,

$$d_{G[H]}(v_{ij}, v_{rs}) = \begin{cases} d_G(v_i, v_r) & \text{if } v_i \neq v_r, \\ a_H(u_j, u_s) & \text{otherwise.} \end{cases}$$

Let S be a subset of $V(G[H])$. The *projection* of S onto H is the set $\{u_j \in V(H) \colon v_{ij} \in S\}$. Also, for $1 \le i \le n$, the i-th *row* of $G[H]$, denoted by R_i, is the set $\{v_{ij} \in V(G[H]) \colon 1 \le j \le m\}$.

Lemma 3.1. *If $W \subseteq V(G[H])$ is a resolving set for $G[H]$, then $W \cap R_i$ resolves R_i, for $1 \le i \le n$. Moreover, the projection of $W \cap R_i$ onto H is an adjacency resolving set for H, for $1 \le i \le n$.*

Proof. Since W resolves $G[H]$, for distinct vertices $v_{ij}, v_{iq} \in R_i$, there exists a vertex $v_{rt} \in W$ such that $d_{G[H]}(v_{rt}, v_{ij}) \neq d_{G[H]}(v_{rt}, v_{iq})$. If $r \neq i$, then $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) = d_{G[H]}(v_{rt}, v_{iq})$, which is a contradiction. Therefore $i = r$ and $W \cap R_i$ resolves R_i.

Now let $u_j, u_q \in V(H)$. Since $W \cap R_i$ resolves R_i, there exists a vertex $v_{it} \in W \cap R_i$ such that $d_{G[H]}(v_{it}, v_{ij}) \neq d_{G[H]}(v_{it}, v_{iq})$. Hence $a_H(u_t, u_j) = d_{G[H]}(v_{it}, v_{ij}) \neq d_{G[H]}(v_{it}, v_{iq}) = a_H(u_t, u_q)$. Consequently the projection of $W \cap R_i$ onto H is an adjacency resolving set for H. $\square$

By Lemma 3.1, every metric basis of $G[H]$ contains at least $\hat{\mu}(H)$ vertices from each copy of H in $G[H]$. Thus the following lower bound for $\mu(G[H])$ is obtained.

Lemma 3.2. *Let G be a connected graph of order n and H be an arbitrary graph. Then*

$$\mu(G[H]) \ge n\hat{\mu}(H).$$

Theorem 3.3. *Let G be a connected graph of order n and H be an arbitrary graph. If there exist two adjacency bases W_1 and W_2 of H such that there is no vertex with adjacency representation **1** with respect to W_1 and no vertex with adjacency representation **2** with respect to W_2, then $\mu(G[H]) = \mu(G[\overline{H}]) = n\hat{\mu}(H)$.*

Proof. By Lemma 3.2, we have $\mu(G[H]) \ge n\hat{\mu}(H)$. To prove the equality, it is enough to provide a resolving set for $G[H]$ of size $n\hat{\mu}(H)$. Let

$$S = \{v_{ij} \in V(G[H]) \colon v_i \in K(G),\ u_j \in W_1\} \cup \{v_{ij} \in V(G[H]) \colon v_i \notin K(G),\ u_j \in W_2\},$$

where $K(G)$ is the set of all vertices of G in equivalence classes of type K. We show that S is a resolving set for $G[H]$. Let $v_{rt}, v_{pq} \in V(G[H]) \setminus S$ be two distinct vertices. The following possibilities can occur.

Case 1. $r = p$. Note that $v_{rt} \neq v_{pq}$ implies $t \neq q$. Since W_1 and W_2 are adjacency resolving sets, there exist vertices $u_j \in W_1$ and $u_l \in W_2$ such that $a_H(u_t, u_j) \neq a_H(u_q, u_j)$ and $a_H(u_t, u_l) \neq a_H(u_q, u_l)$. If $v_r \in K(G)$, then $v_{rj} \in S$ and $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) \neq a_H(u_q, u_j) = d_{G[H]}(v_{pq}, v_{rj})$. Similarly, if $v_r \notin K(G)$, then $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

Case 2. $r \neq p$ and $v_r, v_p \in K(G)$. If v_r and v_p are not twins, then there exists a vertex $v_i \in V(G) \setminus \{v_r, v_p\}$ that is adjacent to only one of the vertices v_r and v_p. Hence for each $u_j \in W_1$, we have $v_{ij} \in S$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. If v_r and v_p are twins, then $v_r \sim v_p$, because $v_r, v_p \in K(G)$. Since $\hat{r}_{W_1}(u_t) \neq \mathbf{1}$, there exists a vertex $u_l \in W_1$ such that $a_H(u_t, u_l) = 2$. Therefore $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) = a_H(u_t, u_l) = 2$. On the other hand, $d_{G[H]}(v_{pq}, v_{rl}) = d_G(v_p, v_r) = 1$. Thus $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

Case 3. $r \neq p, v_r \in K(G)$, and $v_p \notin K(G)$. In this case, v_r and v_p are not twins. Therefore there exists a vertex $v_i \in V(G) \setminus \{v_r, v_p\}$ that is adjacent to only one of the vertices v_r and v_p. Let u_j be a vertex of $W_1 \cup W_2$, such that $v_{ij} \in S$. Hence $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 4. $r \neq p$ and $v_r, v_p \notin K(G)$. If v_r and v_p are not twins, then there exists a vertex $v_i \in V(G) \setminus \{v_r, v_p\}$ that is adjacent to only one of the vertices v_r and v_p. Thus for each $u_j \in W_2$, we have $v_{ij} \in S$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. If v_r and v_p are twins, then $v_r \nsim v_p$, because $v_r, v_p \notin K(G)$. Since $\hat{r}_{W_2}(u_t) \neq \mathbf{2}$, there exists a vertex $u_l \in W_2$, such that $a_H(u_t, u_l) = 1$. Therefore $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) = a_H(u_t, u_l) = 1$. On the other hand, $d_{G[H]}(v_{pq}, v_{rl}) = d_G(v_p, v_r) = 2$, since v_r and v_p are non-adjacent twins in the connected G. Hence $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

Thus $r_S(v_{rt}) \neq r_S(v_{pq})$. Therefore S is a resolving set for $G[H]$ with cardinality $n\hat{\mu}(H)$.

Clearly, in $\overline{H}$, for each $u \in V(\overline{H})$, $\hat{r}_{W_1}(u) \neq \mathbf{2}$ and $\hat{r}_{W_2}(u) \neq \mathbf{1}$. Since $\hat{\mu}(H) = \hat{\mu}(\overline{H})$, by interchanging the roles of W_1 and W_2 for $\overline{H}$, we conclude $\mu(G[\overline{H}]) = n\hat{\mu}(\overline{H}) = n\hat{\mu}(H)$. $\square$

Example 3.4. Label C_5 as $[v_1, \ldots, v_5]$. Note that $\{v_1, v_2\}$ is an adjacency basis such that there is no vertex v in C_5 with $r_{\{v_1,v_2\}}(v) = 1$. Similarly, $\{v_1, v_3\}$ is an adjacency basis such that there is no vertex v in C_5 with $r_{\{v_1,v_3\}}(v) = 2$. By Theorem 3.3, $\mu(G[C_5]) = \mu(G[\overline{C}_5]) = 2n$ for every n-vertex graph G.

In the following three theorems, we obtain $\mu(G[H])$ when H does not satisfy the assumption of Theorem 3.3.

Theorem 3.5. *Let G be a connected graph of order n and H be an arbitrary graph. If for each adjacency basis W of H there exist vertices with adjacency representations $\mathbf{1}$ and $\mathbf{2}$ with respect to W, then $\mu(G[H]) = \mu(G[\overline{H}]) = n(\hat{\mu}(H) + 1) - \iota(G)$.*

Proof. Let B be a metric basis of $G[H]$, R_i be the i-th row of $G[H]$, and B_i be the projection of $B \cap R_i$ onto H, for $1 \le i \le n$. By Lemma 3.1, the B_i's are adjacency resolving sets for H. Therefore $|B \cap R_i| = |B_i| \ge \hat{\mu}(H)$ for $1 \le i \le n$.

Let $I = \{i\colon |B_i| = \hat{\mu}(H)\}$. We claim that $|I| \le \iota(G)$; otherwise by the pigeonhole principle, there exists a pair of twin vertices $v_r, v_p \in V(G)$ such that $|B_r| = |B_p| = \hat{\mu}(H)$. Since B_r and B_p are adjacency bases of H, by the assumption there are vertices u_t and u_q with adjacency representations $\mathbf{1}$ with respect to B_r and B_p, respectively. Also, there are vertices u'_t and u'_q with adjacency representations $\mathbf{2}$ with respect to B_r and B_p, respectively. Hence for each $u \in B_r$ and $u' \in B_p$, we have $u_t \sim u,\ u_{t'} \nsim u,\ u_q \sim u'$, and $u_{q'} \nsim u'$. If $v_r \sim v_p$, then for each $v_{ij} \in B$ one of the following cases can occur.

Case 1. $i \notin \{r, p\}$. Since v_r and v_p are twins, we have $d_G(v_r, v_i) = d_G(v_p, v_i)$. On the other hand, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i)$. Thus $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 2. $i = p \neq r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$. Since $v_i = v_p \sim v_r$, we have $d_G(v_r, v_i) = 1$. On the other hand $u_j \in B_p$, and hence $a_H(u_q, u_j) = 1$. Therefore $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 3. $i = r \neq p$. Similarly to the previous case, $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j) = 1$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i) = 1$. Consequently $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 4. $i = p = r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j)$. Since, $u_j \in B_p = B_r$, we have $a_H(u_q, u_j) = 1 = a_H(u_t, u_j)$. Thus $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Hence $v_r \sim v_p$ implies that $r_B(v_{rt}) = r_B(v_{pq})$, which is a contradiction. Therefore $v_r \nsim v_p$. Since G is a connected graph, non-adjacent twin vertices v_r and v_p have at least one common neighbor, and thus $d_G(v_r, v_p) = 2$. Consequently, by the same method as the case $v_r \sim v_p$, we can see that $r_B(v_{rt'}) = r_B(v_{pq'})$, which contradicts the assumption that B is a metric basis of $G[H]$. Hence $|I| \le \iota(G)$. On the other hand, every metric basis of $G[H]$ has at least $\hat{\mu}(H) + 1$ vertices in R_i, where $i \notin I$. Therefore

$$\begin{aligned}
\mu(G[H]) &= |B| = \bigcup_{i=1}^{n}(B \cap R_i) \ge |I|\hat{\mu}(H) + (n - |I|)(\hat{\mu}(H) + 1) \\
&= n\hat{\mu}(H) + n - |I| \\
&\ge n(\hat{\mu}(H) + 1) - \iota(G).
\end{aligned}$$

Now let W be an adjacency basis of H. By assumption, there exist vertices $u_1, u_2 \in V(H) \setminus W$ such that u_1 is adjacent to all vertices of W and u_2 is not adjacent to any vertex of W. Also, let $K(G)$ be the set of all classes of type K, and let $N(G)$ be the set of all classes of G of type N in G. Choose a fixed vertex v from v^* for each $v^* \in N(G) \cup K(G)$. We claim that the set

$$S = \{v_{ij} \in V(G[H])\colon u_j \in W\} \cup \{v_{t1}\colon v_t \in \cup_{v^* \in K(G)}(v^* \setminus \{v\})\} \cup \{v_{t2}\colon v_t \in \cup_{v^* \in N(G)}(v^* \setminus \{v\})\}$$

is a resolving set for $G[H]$. Let $v_{rt}, v_{pq} \in V(G[H]) \setminus S$. Hence one of the following cases can occur.

Case 1. $r = p$. Since W is an adjacency basis of H, there exists a vertex $u_j \in W$, such that $a_H(u_q, u_j) \neq a_H(u_t, u_j)$. Therefore $d_{G[H]}(v_{pq}, v_{rj}) = a_H(u_q, u_j) \neq a_H(u_t, u_j) = d_{G[H]}(v_{rt}, v_{rj})$. Consequently $r_S(v_{rt}) \neq r_S(v_{pq})$.

Case 2. $r \neq p$ and v_r, v_p are not twins. Hence there exists a vertex $v_i \in V(G)$ that is adjacent to only one of the vertices v_r and v_p. Thus for each vertex $u_j \in W$, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. This yields $r_S(v_{rt}) \neq r_S(v_{pq})$.

Case 3. v_r and v_p are adjacent twins. Therefore at least one of the vertices v_{r1} and v_{p1}, say v_{r1}, belongs to S. Since $v_{rt} \notin S$, we have $t \neq 1$. Hence there exists a vertex $u_j \in S$ such that $a_H(u_t, u_j) = 2$; otherwise $t = 1$. Consequently $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 2$. On the other hand, $d_{G[H]}(v_{pq}, v_{rj}) = d_G(v_p, v_r) = 1$, because $v_r \sim v_p$. This gives $r_S(v_{rt}) \neq r_S(v_{pq})$.

Case 4. v_r and v_p are non-adjacent twins. In this case, at least one of the vertices v_{r2} and v_{p2}, say v_{r2} belongs to S. Hence $t \neq 2$ and there exists a vertex $u_j \in W$, such that $a_H(u_t, u_j) = 1$; otherwise $t = 2$. Therefore $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 1 \neq 2 = d_G(v_p, v_r) = d_{G[H]}(v_{pq}, v_{rj})$. Thus $r_S(v_{rt}) \neq r_S(v_{pq})$.

Consequently, S is a resolving set for $G[H]$ with cardinality

$$|S| = n\hat{\mu}(H) + a(G) - \iota_K(G) + b(G) - \iota_N(G) = n(\hat{\mu}(H) + 1) - \iota(G).$$

Since each adjacency basis of $\overline{H}$ is an adjacency basis of H, we conclude that $\overline{H}$ satisfies the condition of the theorem. Hence $\mu(G[\overline{H}]) = n(\hat{\mu}(H) + 1) - \iota(G)$ and the proof is complete. □

Example 3.6. Let G be the complete graph K_n and label P_3 as $\langle v_1, v_2, v_3\rangle$. Clearly, all adjacency bases of P_3 are $\{v_1\}$ and $\{v_3\}$. Therefore for each adjacency basis of P_3 there exist vertices with adjacency representations **1** and **2**. Since all vertices of K_n are twins, $\iota(K_n) = 1$. Hence by Theorem 3.5, $\mu(K_n[P_3]) = \mu(K_n[\overline{P}_3]) = 2n - 1$.

Theorem 3.7. *Let G be a connected graph of order n and H be an arbitrary graph. If H has the following properties*

(i) *for each adjacency basis of H there exists a vertex with adjacency representation* **1**,
(ii) *there exists an adjacency basis W of H such that there is no vertex with adjacency representation* **2** *with respect to W*,

then $\mu(G[H]) = n\hat{\mu}(H) + a(G) - \iota_K(G)$.

Proof. Let B be a metric basis of $G[H]$, R_i be the i-th row of $G[H]$, and let B_i be the projection of $B \cap R_i$ onto H, for $1 \leq i \leq n$. By Lemma 3.1, each B_i is an adjacency resolving set for H. Therefore $|B \cap R_i| = |B_i| \geq \hat{\mu}(H)$ for $1 \leq i \leq n$.

Let $I = \{i\colon |B_i| = \hat{\mu}(H)\}$. We claim that $|I| \leq n - a(G) + \iota_K(G)$; otherwise, by the pigeonhole principle there exist adjacent twin vertices $v_r, v_p \in V(G)$, such that $|B_r| = |B_p| = \hat{\mu}(H)$. Since B_r and B_p are adjacency bases of H, by assumption (i) there exist vertices $u_t, u_q \in V(H)$ with adjacency representation **1** with respect to B_r and B_p, respectively. Hence for each $u \in B_r$ and each $u' \in B_p$, we have $u_t \sim u$, and $u_q \sim u'$. Since $v_r \sim v_p$, for each $v_{ij} \in B$ one of the following cases can occur.

Case 1. $i \notin \{r, p\}$. Since v_r and v_p are twins, we have $d_G(v_r, v_i) = d_G(v_p, v_i)$. On the other hand, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i)$. Thus $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 2. $i = p \neq r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$. Since $v_i = v_p \sim v_r$, we have $d_G(v_r, v_i) = 1$. On the other hand, $u_j \in B_p$ and hence $a_H(u_q, u_j) = 1$. Therefore $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 3. $i = r \neq p$. Similarly to the previous case, $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j) = 1$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i) = 1$. Consequently $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Case 4. $i = p = r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j)$. Since $u_j \in B_p = B_r$, we have $a_H(u_q, u_j) = 1 = a_H(u_t, u_j)$. Thus $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Hence $r_B(v_{rt}) = r_B(v_{pq})$, which is a contradiction. Therefore $|I| \leq n - a(G) + \iota_K(G)$. On the other hand, every metric basis of $G[H]$ has at least $\hat{\mu}(H) + 1$ vertices in R_i, where $i \notin I$. Thus

$$\begin{aligned} \mu(G[H]) &= |B| \geq |I|\hat{\mu}(H) + (n - |I|)(\hat{\mu}(H) + 1) \\ &= n\hat{\mu}(H) + n - |I| \\ &\geq n\hat{\mu}(H) + a(G) - \iota_K(G). \end{aligned}$$

Now let $K(G)$ be the set of all classes of type K in G and $v \in v^*$ be a fixed vertex for each class v^* of type K. Also, let $u_1 \in V(H) \setminus W$, such that $\hat{r}_W(u_1) = \mathbf{1}$. Consider

$$S = \{v_{ij} \in V(G[H])\colon u_j \in W\} \cup \{v_{t1}\colon v_t \in \cup_{v^* \in K(G)}(v^* \setminus \{v\})\}$$

and let $v_{rt}, v_{pq} \in V(G[H]) \setminus S$. If v_r and v_p are not non-adjacent twins, then as in the proof of Theorem 3.5 we have $r_S(v_{rt}) \neq r_S(v_{pq})$. Now let v_r and v_p be non-adjacent twin vertices of G. By assumption, there exists a vertex $u_j \in W$, such that $a_H(u_t, u_j) = 1$. Therefore $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 1$. On the other hand, $d_{G[H]}(v_{pq}, v_{rj}) = d_G(v_p, v_r) = 2$, since v_r and v_p are non-adjacent twins in the connected graph G. Hence $r_S(v_{rt}) \neq r_S(v_{pq})$. This implies that S is a resolving set for $G[H]$ with cardinality $n\hat{\mu}(H) + a(G) - \iota_K(G)$. □

Example 3.8. Let G be the bipartite graph $K_{r,s}$, where $r+s \geq 3$ and label C_3 as $[v_1, v_2, v_3]$. Clearly, all adjacency bases of C_3 are $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$. Therefore for each adjacency basis of C_3 there exists a vertex with adjacency representation **1** and no vertex with adjacency representation **2**. Moreover, $a(K_{r,s}) = 0$ and $\iota_K(K_{r,s}) = 0$. Hence by Theorem 3.7, $\mu(K_{r,s}[C_3]) = 2(r+s)$.

By a similar proof, we have the following theorem.

Theorem 3.9. *Let G be a connected graph of order n and H be an arbitrary graph. If H has the following properties*

(i) *for each adjacency basis of H there exists a vertex with adjacency representation* **2**,
(ii) *there exists an adjacency basis W of H such that there is no vertex with adjacency representation* **1** *with respect to W,*

then $\mu(G[H]) = n\hat{\mu}(H) + b(G) - \iota_N(G)$.

Example 3.10. Let G be the complete graph $K_{r,s}$, where $r, s \geq 2$, and let P be the Petersen graph, the graph whose vertices are 2-subsets of a 5-element set and whose edges are the pairs of disjoint 2-subsets. Since $\mathrm{diam}(P) = 2$, $\hat{\mu}(P) = \mu(P) = 3$. It is easy to check that for each adjacency basis of P there exists a vertex with adjacency representation **2** and no vertex with adjacency representation **1** in P. Moreover, $b(K_{r,s}) = r+s$ and $\iota_N(K_{r,s}) = 2$. Hence by Theorem 3.9, $\mu(K_{r,s}[P]) = 4(r+s)-2$.

Corollary 3.11. *If G is a connected graph of order n that has no twin vertices, then $\mu(G[H]) = n\hat{\mu}(H)$.*

Proof. The adjacency bases of H satisfy the conditions of one of Theorems 3.3, 3.5, 3.7 and 3.9. Now if G does not have any pair of twin vertices, then $\iota(G) = n$, $\iota_K(G) = a(G) = 0$, and $\iota_N(G) = b(G) = 0$. Therefore $\mu(G[H]) = n\hat{\mu}(H)$. □

If the adjacency dimension of a graph H is known, then by Theorems 3.3, 3.5, 3.7 and 3.9, the exact value of $\mu(G[H])$ of many graphs G and H can be determined. In the following two corollaries, $\mu(G[H])$ for some well known graphs is obtained.

Corollary 3.12. *Let $G = P_n$, $n \geq 4$ or $G = C_n$, $n \geq 5$ and m be an integer, $m \geq 3$. Then $\mu(G[P_m]) = \mu(G[C_m]) = \mu(G[\overline{P}_m]) = \mu(G[\overline{C}_m]) = n\lfloor \frac{2m+2}{5} \rfloor$. Moreover,*

$$\mu(G[\overline{K}_{m_1,\ldots,m_t}]) = \mu(G[K_{m_1,\ldots,m_t}]) = \begin{cases} n(m-r-1) & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t, \end{cases}$$

where $m_1, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$.

Proof. If $G = P_n$, $n \geq 4$ or $G = C_n$, $n \geq 5$, then G does not have any pair of twin vertices. Thus by Corollary 3.11, $\mu(G[H]) = n\hat{\mu}(H)$, for each graph H. In particular, by Propositions 2.3 and 2.7, $\hat{\mu}(P_m) = \hat{\mu}(C_m) = \hat{\mu}(\overline{P}_m) = \hat{\mu}(\overline{C}_m) = \lfloor \frac{2m+2}{5} \rfloor$. Therefore $\mu(G[P_m]) = \mu(G[C_m]) = \mu(G[\overline{P}_m]) = \mu(G[\overline{C}_m]) = n\lfloor \frac{2m+2}{5} \rfloor$. Also, by Propositions 2.3 and 2.8, we have

$$\mu(G[\overline{K}_{m_1,\ldots,m_t}]) = \mu(G[K_{m_1,\ldots,m_t}]) = \begin{cases} n(m-r-1) & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t. \end{cases} \quad \square$$

Corollary 3.13. *Let $m, n, m_1, \ldots, m_t$ be integers such that $m_1, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, $\sum_{i=1}^{t} m_i = m$, and $n \geq 2$. Then*

$$\mu(K_n[K_{m_1,\ldots,m_t}]) = \begin{cases} n(m-r)-1 & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t. \end{cases}$$

Proof. Let $H = K_{m_1,\ldots,m_t}$. For each adjacency basis of H, there is no vertex of H with adjacency representation **2**. If $r = t$, then for each adjacency basis of H there is no vertex of H with adjacency representation **1**. Therefore by Theorem 3.3, $\mu(G[H]) = n\hat{\mu}(H)$ for each connected graph G of order n. If $r \neq t$, then for each adjacency basis of H, there exists a vertex with adjacency representation **1**. Thus by Theorem 3.7, $\mu(G[H]) = n\hat{\mu}(H) + a(G) - \iota_K(G)$ for each connected graph G of order n. Therefore by Proposition 2.8, we have the desired equality. □

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