Discrete Mathematics 232 (2001) 175–183
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Chromatic equivalence classes of certain cycles with edges ☆

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Received 7 November 1997; revised 26 March 1999; accepted 14 August 2000

Abstract

Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written $G{\sim}H$, if $P(G)=P(H)$. A graph G is chromatically unique if for any graph H, $G{\sim}H$ implies that G is isomorphic with H. In this paper, we give the necessary and sufficient conditions for a family of generalized polygon trees to be chromatically unique. © 2001 Elsevier Science B.V. All rights reserved.

MSC: primary 05C15

Keywords: Generalized polygon trees; Chromatically unique graph; Chromatic equivalence class

1. Introduction

The graphs that we consider are finite, undirected and simple. Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are said to be *chromatically equivalent*, and we write $G{\sim}H$, if $P(G)=P(H)$. A graph G is *chromatically unique* if G is isomorphic with H for any graph H such that $G \sim H$. A set of graphs $\mathscr{S}$ is called a *chromatic equivalence class* if for any graph H, that is chromatically equivalent with a graph G in $\mathscr{S}$, $H \in \mathscr{S}$.

A path in G is called a *simple* path if the degree of each interior vertex is two in G. A *generalized polygon tree* is a graph defined recursively as follows. A cycle C_p ($p \geqslant 3$) is a generalized polygon tree. Next, suppose H is a generalized polygon tree containing a simple path P_k, where $k \geqslant 1$. If G is a graph obtained from the union of H and a cycle C_r, where $r > k$, by identifying P_k in H with a path of length k in

☆ This work was partly supported by the University Putra Malaysia research grant # 50438-97-09.
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PII: S0012-365X(00)00355-1

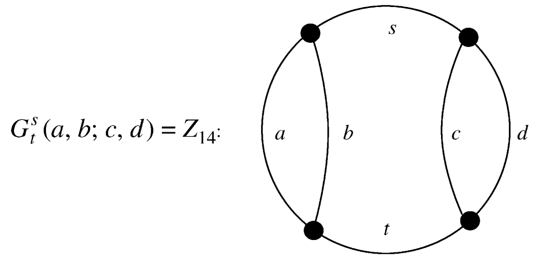

Fig. 1. $G_t^s(a,b;c,d)$, $s,t \geqslant 0$.

C_r, then G is also a generalized polygon tree. Consider the generalized polygon tree $G_t^s(a,b;c,d)$ with three interior regions shown in Fig. 1. The integers a,b,c,d,s and t represent the lengths of the respective paths between the vertices of degree three, where $s \geqslant 0$, $t \geqslant 0$. Without loss of generality, assume that $a \leqslant b$ and $a \leqslant c \leqslant d$. Thus, $\min\{a,b,c,d\} = a$. Let $r = s + t$. We now form a family $\mathscr{C}_r(a,b;c,d)$ of the graphs $G_t^s(a,b;c,d)$ where the values of a, b, c, d and r are fixed but the values of s and t vary; that is

$$\mathscr{C}_r(a,b;c,d) = \{G_t^s(a,b;c,d) \mid r = s + t,\ s \geqslant 0,\ t \geqslant 0\}.$$

It is clear that the families $\mathscr{C}_0(a,b;c,d)$ and $\mathscr{C}_1(a,b;c,d)$ are singletons.

In [1], Chao and Zhao studied the chromatic polynomials of the family $\mathscr{F}$ of connected graphs with k edges and $(k-2)$ vertices each of whose degrees is at least two, where k is at least six. They first divided this family of graphs into three subfamilies $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ according to their chromatic polynomials, and computed the chromatic polynomials for the graphs in each subfamily. Then they discussed the chromatic equivalence of graphs in $\mathscr{F}$, and proved many results. One of these results is Theorem B which is stated at the end of this section. They also discussed the chromatic uniqueness of graphs in $\mathscr{F}_3$ but they did not study the chromatic uniqueness of graphs in $\mathscr{F}_2$ which consists of graphs of types Z_{12}, Z_{13} and Z_{14}. Note that the graph $G_t^s(a,b;c,d)$ is in $\mathscr{F}_2$. In fact, $G_0^0(a,b;c,d) = Z_{12}$, the graph $G_t^s(a,b;c,d)$ shown in Fig. 1 is the graph Z_{14} where $s + t = j_1 + j_2$, and $G_r^0(a,b;c,d)$ with $r \geqslant 1$ is exactly the graph Z_{13}, where $j = r$. On the other hand $\mathscr{C}_r(a,b;c,d) = \mathscr{F}_2$.

Xu et al. [5] gave the necessary and sufficient conditions for $G_0^0(a,b;c,d)$ to be chromatically unique. In their paper, they called $G_0^0(a,b;c,d)$ a 4-bridge graph. In [2], Peng showed that the graph $G_1^0(a,b;c,d)$ is chromatically unique if each of the a, b, c, and d is at least four. Note that if $r \geqslant 2$, then $G_r^0(a,b;c,d)$ is not a chromatically unique graph and it is clear that for each $r \geqslant 1$, the graph $G_r^0(a,b;c,d)$ with $\min\{a,b,c,d\} = 1$ is not chromatically unique. In this paper, we characterize the chromaticity of $G_1^0(a,b;c,d)$ for a, b, c or d less than four.

In the remaining of this section, we state some known results that will be used to prove our main theorems. The girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle of G.

Theorem A (Whitney [4]). *Let G and H be chromatically equivalent graphs. Then*

(a) $|V(G)| = |V(H)|$,
(b) $|E(G)| = |E(H)|$,
(c) $g(G) = g(H)$,
(d) *G and H have the same number of shortest cycles.*

Theorem B (Chao and Zhao [1], Peng et al. [3]). *All the graphs in $\mathscr{C}_r(a,b;c,d)$ are chromatically equivalent.*

By this theorem we only need to compute $P(G_r^0(a,b;c,d))$ for computing the chromatic polynomial of $G_t^s(a,b;c,d)$

Theorem C (Peng [2]). *If $G_1^0(a,b;c,d)$ and $G_1^0(a',b';c',d')$ are chromatically equivalent, then they are isomorphic.*

The next known result gives the chromatic polynomial of $G_t^s(a,b;c,d)$. In [1], Chao and Zhao also determined the chromatic polynomial of this graph, but we shall use the computed chromatic polynomial of $G_t^s(a,b;c,d)$, $s,t \geqslant 0$ in [3] to prove our main results.

Theorem D (Peng et al. [3]). *Let the order of $G_t^s(a,b;c,d)$ be n ($n = a+b+c+d+r-2$), and $x = 1-\lambda$. Then we have*

$$P(G_t^s(a,b;c,d)) = \frac{(-1)^n x}{(x-1)^2} \cdot Q(G_t^s(a,b;c,d)),$$

where

$$\begin{aligned} Q(G_t^s(a,b;c,d)) = {} & (x^{n+1} - x^{a+b+r} - x^{c+d+r} + x^{r+1} - x) \\ & -(1+x+x^2) + (x+1)(x^a + x^b + x^c + x^d) \\ & -(x^{a+c} + x^{a+d} + x^{b+c} + x^{b+d}). \end{aligned}$$

2. Main results

In this section, we shall characterize the chromaticity of $G_1^0(a,b;c,d)$ when $\min\{a,b,c,d\} < 4$. First, we consider the case when $\min\{a,b,c,d\} = 2$. In Theorem 2, we consider the case when $\min\{a,b,c,d\} = 3$.

Theorem 1. *The graph $G_1^0(a,b;c,d)$ when $\min\{a,b,c,d\} = 2$ is chromatically unique if and only if $G_1^0(a,b;c,d)$ is not isomorphic with $G_1^0(2,3;3,5)$.*

Proof. Let $G = G_1^0(a,b;c,d)$ and $H \sim G$. By Lemma 4 and Theorem 2 in [1], we have $H = G_{t'}^{s'}(a',b';c',d')$, where a',b',c',d' are at least two. If $r' = 1$ then by Theorem C,

$G \cong H$. Now suppose that $r' \geqslant 2$. We solve the equation $Q(G)=Q(H)$. After cancelling the terms x^{n+1}, $-x$ and $-(1+x+x^2)$, we have $Q_1(G)=Q_1(H)$ where

$$Q_1(G)=x^2+(x+1)(x^a+x^b+x^c+x^d)-x^{1+a+b}-x^{1+c+d}$$
$$-x^{a+c}-x^{a+d}-x^{b+c}-x^{b+d},$$
$$Q_1(H)=x^{r'+1}+(x+1)(x^{a'}+x^{b'}+x^{c'}+x^{d'})-x^{r'+a'+b'}$$
$$-x^{r'+c'+d'}-x^{a'+c'}-x^{a'+d'}-x^{b'+c'}-x^{b'+d'}$$

and

$$a+b+c+d+1=a'+b'+c'+d'+r'.$$

Without loss of generality, assume that $a \leqslant b$, $a \leqslant c \leqslant d$, and $a' \leqslant b'$, $a' \leqslant c' \leqslant d'$. It is easy to see that $\min\{a,b,c,d,2\}=\min\{a',b',c',d',r'+1\}$. This means $2=\min\{a',r'+1\}$. If $r'+1=2$, then $r'=1$ and this contradicts our assumption; thus $a'=2$. Also we have $2=a=\min\{a,b,c,d\}=\min\{r'+1,b',c',d'\}$ and we know that $r'+1\neq 2$. Therefore, $b'=2$ or $c'=2$. We now consider these two cases.

Case 1: Suppose $b'=2$. Then from $Q_1(G)=Q_1(H)$, after cancelling equal terms, we have $Q_2(G)=Q_2(H)$ where

$$Q_2(G)=(x+1)(x^b+x^c+x^d)-x^{3+b}-x^{1+c+d}$$
$$-x^{2+c}-x^{2+d}-x^{b+c}-x^{b+d},$$
$$Q_2(H)=x^{r'+1}+(x+1)(x^{c'}+x^{d'})+x^3-x^{r'+4}$$
$$-x^{r'+c'+d'}-x^{2+c'}-x^{2+d'}-x^{2+c'}-x^{2+d'}$$

and

$$b+c+d=c'+d'+r'+1; \qquad a=2\leqslant b,\ 2\leqslant c\leqslant d, \quad a'=b'=2,\ 2\leqslant c'\leqslant d'.$$

Since $a'=b'=2$, $g(G)=g(H)=4$. Therefore, $b=2$ or $c=d=2$ because $a=2$.

Subcase 1.1: Suppose $b=2$. Then $x^2\in Q_2(G)$ and x^2 cannot be cancelled in $Q_2(G)$. So we must have $x^2\in Q_2(H)$. Hence $r'+1=2$ or $c'=2$. But $r'+1=2$ contradicts our assumption. Therefore we have $c'=2$ and $Q_3(G)=Q_3(H)$, where

$$Q_3(G)=(x+1)(x^c+x^d)-x^5-x^{1+c+d}-2x^{2+c}-2x^{2+d},$$
$$Q_3(H)=x^{r'+1}+(x+1)(x^{d'})+x^3-x^{r'+4}-x^{r'+d'+2}-2x^4-2x^{2+d'}$$

and

$$c+d=d'+r'+1; \qquad a=b=2, \quad 2\leqslant c\leqslant d, \quad a'=b'=2, \quad 2=c'\leqslant d'.$$

Since $x^3\in Q_3(H)$ and cannot be cancelled, we must have $x^3\in Q_3(G)$. Thus $c=3$ or $d=3$ or $c+1=3$ or $d+1=3$. If $d=3$, then we have $c=2$ or $c=3$ because $d\geqslant c\geqslant 2$, and similarly if $d+1=3$ (or $d=2$), then $c=2$. Hence, it is sufficient to consider two cases when $c+1=3$ or $c=3$.

Subcase 1.1.1: Suppose $c=3$. Since $x^4 \in Q_3(G)$ and cannot be cancelled, and since $-2x^4 \in Q_3(H)$, we must have $3x^4 \in Q_3(H)$. But $r'+1=d'=d'+1=4$, which is impossible.

Subcase 1.1.2: Suppose $c+1=3$ (or $c=2$). Then $x^2 \in Q_3(G)$ and cannot be cancelled. Since $r'+1 \neq 2$, we have $d'=2$. This means H has two cycles of shortest length but G has only one cycle of the shortest length because $d=r'+1 \neq 2$.

The two subcases above show that $b=2$ is impossible.

Subcase 1.2: Suppose $c=d=2$ and $b \neq 2$. Then $g(G)=4$ and G has only one cycle of the shortest length. By Theorem A, H must have only one cycle of the shortest length; therefore $d' \neq 2$. Then from $Q_2(G)=Q_2(H)$, after cancelling equal terms, we have $Q_4(G)=Q_4(H)$, where

$$Q_4(G)=(x+1)x^b+2x^2+2x^3-x^{3+b}-x^5-2x^4-2x^{2+b},$$

$$Q_4(H)=x^{r'+1}+(x+1)(x^{c'}+x^{d'})+x^3-x^{r'+4}-x^{r'+c'+d'}-2x^{2+c'}-2x^{2+d'}$$

and

$$b+3=c'+d'+r'; \qquad a=c=d=2, \quad 2\leqslant b, \quad a'=b'=2, \quad 2\leqslant c'\leqslant d'.$$

Since $2x^2 \in Q_4(G)$ and cannot be cancelled, we must have $2x^2 \in Q_4(H)$. But this is impossible because $r'+1 \neq 2$ and $d' \neq 2$. So we have no solution for $Q(G)=Q(H)$ when $b'=2$.

Case 2: Suppose $c'=2$. Then from $Q_1(G)=Q_1(H)$, after cancelling equal terms, we have $Q_5(G)=Q_5(H)$, where

$$Q_5(G)=(x+1)(x^b+x^c+x^d)-x^{3+b}-x^{1+c+d}-x^{2+c}-x^{2+d}-x^{b+c}-x^{b+d},$$

$$Q_5(H)=x^{r'+1}+(x+1)(x^{b'}+x^{d'})+x^3-x^{r'+b'+2}$$
$$-x^{r'+d'+2}-x^4-x^{2+d'}-x^{b'+d'}-x^{b'+2},$$

and

$$b+c+d=b'+d'+r'+1; \quad a=2\leqslant b, \quad 2\leqslant c\leqslant d, \quad a'=2\leqslant b', \quad 2=c'\leqslant d'.$$

Since $a'=c'$, without loss of generality, we assume $b'\leqslant d'$. From Case 1, $b' \neq 2$; therefore $g(G)=g(H)>4$ and $b\geqslant 3$. Since $x^3 \in Q_5(H)$ and cannot be cancelled, we must have $x^3 \in Q_5(G)$ and thus $b=3$ or $c=3$. The case $c=2$ and the case $d=2$ are impossible because $x^2 \notin Q_5(H)$. ($r'+1 \neq 2$, $b' \neq 2$ and $b'\leqslant d'$.) Also the case $d=3$ implies that $c=2$ or $c=3$. We now consider cases when $b=3$ and $c=3$.

Subcase 2.1: Suppose $b=3$. Then $g(G)=g(H)=5$. Therefore, $b'=3$ because $g(H)=a'+b'=2+b'$. Now we have $Q_6(G)=Q_6(H)$, where

$$Q_6(G)=(x+1)(x^c+x^d)-x^6-x^{1+c+d}-x^{2+c}-x^{2+d}-x^{3+c}-x^{3+d},$$

$$Q_6(H)=x^{r'+1}+(x+1)x^{d'}+x^3-x^{r'+5}-x^{r'+d'+2}-x^4-x^{2+d'}-x^5-x^{3+d'}$$

and

$$c+d=d'+r'+1; \quad a=2, \quad b=3, \quad 3\leqslant c\leqslant d, \quad a'=2,$$
$$b'=3, \quad c'=2, \quad 3\leqslant d'.$$

Since $x^3\in Q_6(H)$ and cannot be cancelled, $x^3\in Q_6(G)$ and so we have $c=3$. We now have $Q_7(G)=Q_7(H)$, where

$$Q_7(G)=(x+1)x^d+x^4-x^6-x^{4+d}-x^5-x^{2+d}-x^6-x^{3+d},$$
$$Q_7(H)=x^{r'+1}+(x+1)x^{d'}-x^{r'+5}-x^{r'+d'+2}-x^4-x^{2+d'}-x^5-x^{3+d'}$$

and

$$2+d=d'+r'.$$

Since $3=c\leqslant d$, $d\neq 2$ and thus x^4 in $Q_7(G)$ cannot be cancelled. So we must have $2x^4\in Q_7(G)$ because $-x^4\in Q_7(H)$. This means we have either $r'=3$ and $d'=4$ or $r'=3$ and $d'=3$. If the former holds, then $d=5$ and we get one solution for $Q(G)=Q(H)$, that is $a=2, b=c=3$ and $d=5$; also $a'=2, b'=3, c'=2, d'=4$ and $r'=3$. With these values we have $G_1^0(2,3;\ 3,5)\sim G_3^0(2,3;\ 2,4)$ but $G_1^0(2,3;\ 3,5)\not\cong G_3^0(2,3;\ 2,4)$. If the latter holds, then $d=4$ and we have $Q_8(G)=Q_8(H)$, where

$$Q_8(G)=x^4-x^6-x^8-x^6-x^6-x^7,$$
$$Q_8(H)=x^3-x^8-x^8-x^5-x^5-x^6$$

and it is a contradiction.

Subcase 2.2: Suppose $c=3$ and $b\neq 3$. Then $g(G)=6=g(H)$. Since $b'\leqslant d'$, we have $r'=2$ or $b'=4$. If the former holds, then from $Q_5(G)=Q_5(H)$, after cancelling equal terms, we have $Q_9(G)=Q_9(H)$ where

$$Q_9(G)=(x+1)(x^b+x^d)+x^4-x^{3+b}-x^{4+d}-x^5-x^{2+d}-x^{3+b}-x^{b+d},$$
$$Q_9(H)=x^3+(x+1)(x^{b'}+x^{d'})-x^{4+b'}-x^{4+d'}-x^4-x^{2+d'}-x^{b'+d'}-x^{b'+2}$$

and

$$b+d=b'+d'; \quad a=2\leqslant b, \quad 3=c\leqslant d, \quad r'=2, \quad a'=2, \quad 4\leqslant b',$$
$$c'=2, \quad 4\leqslant d'.$$

Now $x^3\in Q_9(H)$ and cannot be cancelled. Therefore, $x^3\in Q_9(G)$; hence, $d=3$ because $b\neq 3$. With this we have $2x^4\in Q_9(G)$ and cannot be cancelled. Since $-x^4\in Q_9(H)$, we must have $3x^4\in Q_9(H)$, and this is impossible. If the latter holds, then from $Q_5(G)=Q_5(H)$, after cancelling equal terms, we have $Q_{10}(G)=Q_{10}(H)$, where

$$Q_{10}(G)=(x+1)(x^b+x^d)-x^{3+b}-x^{4+d}-x^5-x^{2+d}-x^{3+b}-x^{b+d},$$
$$Q_{10}(H)=x^{r'+1}+(x+1)x^{d'}+x^5-x^{r'+6}-x^{r'+d'+2}-x^4-x^{2+d'}-x^{4+d'}-x^6$$

and

$$b+d=d'+r'+2; \quad a=2\leqslant b, \quad 3=c\leqslant d, \quad a'=2, \quad b'=4, \quad c'=2, \quad 4\leqslant d'.$$

Now $x^5 \in Q_{10}(H)$ and cannot be cancelled. Since $-x^5 \in Q_{10}(G)$, we must have $2x^5 \in Q_{10}(G)$. If $b{=}d{=}5$, then $2x^6 \in Q_{10}(G)$ and cannot be cancelled, and since $-x^6 \in Q_{10}(H)$ we must have $3x^6 \in Q_{10}(H)$; and this is impossible. If $b=d=4$, then $2x^4 \in Q_{10}(G)$ cannot be cancelled and since $-x^4 \in Q_{10}(H)$, we must have $3x^4 \in Q_{10}(H)$, and this is impossible. If $b{=}4$ and $d{=}5$, then $x^6 \in Q_{10}(G)$ and cannot be cancelled and since $-x^4$ and $-x^6$ are in $Q_{10}(H)$, we must have $2x^4$ and $2x^6$ in $Q_{10}(H)$ which is impossible. If $b=5$ and $d=4$, then we have $Q_{11}(G)=Q_{11}(H)$ where

$$Q_{11}(G)=x^4-x^8-x^8-x^8-x^9,$$

$$Q_{11}(H)=x^{r'+1}+(x+1)x^{d'}-x^{r'+6}-x^{r'+d'+2}-x^4-x^{2+d'}-x^{4+d'}-x^6$$

and

$$7=d'+r'; \quad 4\leqslant d'.$$

Since $-x^4$ and $-x^6$ are in $Q_{11}(H)$ but they are not in $Q_{11}(G)$ and since $x^4 \in Q_{11}(G)$ cannot be cancelled in $Q_{11}(G)$, we must have $2x^4$ and x^6 in $Q_{11}(H)$, but this is impossible. Therefore $Q(G)=Q(H)$ has no other solution when $c'=2$. □

The next main result is for the case when $\min\{a,b,c,d\}=3$. The proof is similar to that of Theorem 1. The detailed proof can be obtained by e-mail from the second author or view at `http://www.fsas.upm.edu.my/~yhpeng/publish/prooft2.pdf`

Theorem 2. *The graph* $G_1^0(a,b;\ c,d)$ *when* $\min\{a,b,c,d\}=3$ *is chromatically unique if and only if* $G_1^0(a,b;\ c,d)$ *is not isomorphic with* $G_1^0(3,b;\ b+1,b+3)$ *and* $G_1^0(3,c+3;\ c,c+1)$ *and* $G_1^0(3,3;\ c,c+2)$ *and* $G_1^0(3,b;\ 3,b+2)$ and $G_1^0(3,5;\ 5,8)$.

The following theorem follows from the proof of Theorems 1 and 2.

Theorem 3. *Each of the following families is a chromatic equivalence class.*

(a) $\mathscr{C}_1(2,3;\ 3,5)\cup\mathscr{C}_3(2,3;\ 2,4)$.
(b) $\mathscr{C}_1(3,5;\ 5,8)\cup\mathscr{C}_5(2,6;\ 4,5)$.
(c) $\mathscr{C}_1(3,b;\ b+1,b+3)\cup\mathscr{C}_3(2,b+1;\ b,b+2)$ *for any* $b\geqslant 3$.
(d) $\mathscr{C}_1(3,b+3;\ b,b+1)\cup\mathscr{C}_3(2,b+2;\ b,b+1)$ *for any* $b\geqslant 3$.
(e) $\mathscr{C}_1(3,3;\ b,b+2)\cup\mathscr{C}_{b-1}(2,4;\ 3,b+1)$ *for any* $b\geqslant 3$.
(f) $\mathscr{C}_1(3,b;\ 3,b+2)\cup\mathscr{C}_{b-1}(2,b+1;\ 3,4)$ *for any* $b\geqslant 3$.

Remark. Note that if $b=2$ in the graphs (c) and (d), then we get the graph (a).

Corollary. *Each of the following families of graphs is not a chromatic equivalence class.*

(a) $\mathscr{C}_5(2,6;\ 4,5)$.
(b) $\mathscr{C}_3(2,b+1;\ b,b+2)$ $(b\geqslant 2)$.
(c) $\mathscr{C}_3(2,b+2;\ b,b+1)$ $(b\geqslant 2)$.
(d) $\mathscr{C}_r(2,4;\ 3,r+2)$ $(r\geqslant 2)$.
(e) $\mathscr{C}_r(2,r+2;\ 3,4)$ $(r\geqslant 2)$.

Combining Theorem 3 in [2] and Theorems 1 and 2 above, we have the following characterization theorem.

Theorem 4. *The graph* $G_1^0(a,b;\ c,d)$ *with* $\min\{a,b,c,d\} > 1$ *is chromatically unique if and only if* $G_1^0(a,b;\ c,d)$ *is not isomorphic with any one of the following graphs.*

(a) $G_1^0(2,3;\ 3,5)$,
(b) $G_1^0(3,5;\ 5,8)$,
(c) $G_1^0(3,b;\ b+1,b+3)$ *for any* $b\geqslant 3$,
(d) $G_1^0(3,c+3;\ ,c,c+1)$ *for any* $c\geqslant 3$,
(e) $G_1^0(3,3;\ c,c+2)$ *for any* $c\geqslant 3$,
(f) $G_1^0(3,b;\ 3,b+2)$ *for any* $b\geqslant 3$.

Remark. Note that if $b=2$ in the graph (c) and if $c=2$ in the graph (d), then we get the graph (a).

We also discover that the conjecture in [3] is only true for $r=1$. For each $r\geqslant 2$, we provide two counter examples as follows:

- $G_r^0(r+2,b;\ b+1,b+r+2) \sim G_{r+2}^0(r+1,b+1;\ b,b+r+1)$ for $b\geqslant 4$ but $G_{r+2}^0(r+1,b+1;\ b,b+r+1) \notin \mathscr{C}_r(r+2,b;\ b+1,b+r+2)$.
- $G_r^0(r+2,c+r+2;\ c,c+1) \sim G_{r+2}^0(r+1,c+r+1;\ c,c+1)$ for $c\geqslant 4$ but $G_{r+2}^0(r+1,c+r+1;\ c,c+1) \notin \mathscr{C}_r(r+2,c+r+2;\ c,c+1)$.

We discuss the chromatic equivalence of graphs in $\mathscr{C}_r(a,b;\ c,d)$ $(r\geqslant 2)$ in another article.

Acknowledgements

The authors wish to thank the referees who made many suggestions, and shortened our manuscript a great deal.

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