Proof: Let $(u,v) \in E(G)$. Since $g(G) \geq 4$ we have $N(u) \cap N(v) = \emptyset$. Then there are at most $2k(k-1)+2$ vertices at distance at most three from at least one of the vertices u and v. Since $e(G) = \frac{nk}{2}$ it follows that there are at least $\frac{nk}{2[2(k-1)[k(k-1)+1]]}$ edges whose neighbors are disjoint (since $g(G) \geq 4$). Label $st(u) = st(v) = 1$. Then $2k$ vertices (including u and v) are dominated. Hence, running over all such edges we have that at least $\frac{nk}{2[2(k-1)[k(k-1)+1]]} \bullet 2k$ vertices are dominated. The rest of the vertices are labeled 0.

Hence,

$$\gamma_S(G;0,1) \leq n - k\frac{nk}{2(k-1)[k(k-1)+1]} + \frac{nk}{2(k-1)[k(k-1)+1]} = n - n\frac{k(k-1)}{2(k-1)[k(k-1)+1]} = n\left(1 - \frac{k}{2[k(k-1)+1]}\right) \leq n(1 - \frac{1}{2k}).$$

■

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Chromatic Equivalence Classes of Certain Generalized Polygon Trees, II

BEHNAZ OMOOMI [a] and YEE-HOCK PENG [b1]

[a] *Department of Mathematical Sciences Isfahan University of Technology 84154, Isfahan, Iran*

[b] *Department of Mathematics, and Institute for Mathematical research University Putra Malaysia 43400UPM Serdang, Malaysia*

ABSTRACT

Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written $G \sim H$, if $P(G) = P(H)$. A graph G is chromatically unique if for any graph H, $G \sim H$ implies that G is isomorphic with H. In "Chromatic Equivalence Classes of Certain Generalized Polygon Trees", Discrete Mathematics Vol.172, 103 - 114 (1997), Peng et al. studied the chromaticity of certain generalized polygon trees. In this paper, we present a chromaticity characterization of another big family of such graphs.

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1. Introduction

The graphs that we consider are finite, undirected and simple. Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are said to be *chromatically equivalent*, and we write $G \sim H$, if $P(G) = P(H)$. A graph G is *chromatically unique* if $G \sim H$ implies that H is isomorphic to G. A set of graphs S is called a *chromatic equivalence class* if any two element of S are

[1] Corresponding author. E-mail: yhpeng@fsas.upm.edu.my

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chromatically equivalent, and if any graph which is chromatically equivalent with a graph G in S is also isomorphic to some element of S. Although chromatically unique graphs have been the subject of many recent papers (see [2] and [3]), relatively few results concerning the chromatic equivalence classes of graphs are known.

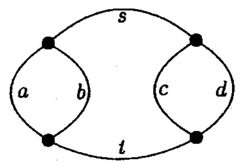

Figure 1. $G_t^s(a,b;c,d)$

A path in G is called *simple* if the degree of each interior vertex is two in G. A ***generalized polygon tree*** is a graph defined recursively as follows. A cycle C_p ($p \geq 3$) is a generalized polygon tree. Next, suppose H is a generalized polygon tree containing a simple path P_k, where $k \geq 1$. If G is a graph obtained from the union of H and a cycle C_r, where $r > k$, by identifying P_k in H with a path of length k in C_r, then G is also a generalized polygon tree. Consider the generalized polygon tree $G_t^s(a,b;c,d)$ shown in Figure 1. The integers a, b, c, d, s and t represent the lengths of the respective paths between the vertices of degree > 2, where $s \geq 0$, $t \geq 0$. Without loss of generality, assume that $a \leq b$, $a \leq c \leq d$ and if $a = c$, then $b \leq d$. Thus, $\min\{a,b,c,d\} = a$. Let $r = s+t$. We now form a family $\mathcal{C}_r(a,b;c,d)$ of the graphs $G_t^s(a,b;c,d)$ where the values of a, b, c, d and r are fixed but the values of s and t vary; that is

$$\mathcal{C}_r(a,b;c,d) = \{\, G_t^s(a,b;c,d) \mid r = s+t, s \geq 0, t \geq 0 \,\}.$$

It is clear that the families $\mathcal{C}_0(a,b;c,d)$ and $\mathcal{C}_1(a,b;c,d)$ are singletons.

Note that $G_t^s(a,b;c,d)$ is a connected $(n,n+2)$-graph, whose chromatic polynomials were computed by Chao and Zhao (see [1]), who also determined several chromatic equivalence classes, excluding among others the graph $G_t^s(a,b;c,d)$.

In [6], Peng et al. showed that $\mathcal{C}_r(a,b;c,d)$ is a chromatic equivalence class for a, b, c, d at least $r+3$. As a corollary, the graph $G_1^0(a,b;c,d)$ is chromatically unique for a, b, c, d at least four (see also Peng [5]). In [4], Omoomi and Peng characterized the chromaticity of $\mathcal{C}_1(a,b;c,d)$ for the minimum a,b,c, and d

less than four. In this paper, we characterize the chromaticity of $\mathcal{C}_r(a,b;c,d)$ for the minimum of a, b, c and d equal to $r+2$, and $r \geq 2$. Also we discover that the following conjecture is not true for each $r \geq 2$.

Conjecture [6]. *The family of graphs $\mathcal{C}_r(a,b;c,d)$ is a chromatic equivalence class whenever a,b,c, and d are each at least four.*

In the remaining of this section, we give some known results that will be used to prove our main theorem. The girth of G, denoted by $g(G)$, is the length of a shortest cycle of G.

Theorem A (Whitney [7]). *Let G and H be chromatically equivalent graphs. Then*

(a) $|V(G)| = |V(H)|$.

(b) $|E(G)| = |E(H)|$.

(c) $g(G) = g(H)$.

(d) *G and H have the same number of shortest cycles.*

Theorem B (Chao and Zhao [1], Peng et al. [6]). *All the graphs in $\mathcal{C}_r(a,b;c,d)$ are chromatically equivalent.*

By this theorem we only need to compute $P(G_r^0(a,b;c,d))$ for computing the chromatic polynomial of $G_t^s(a,b;c,d)$.

The next result is Case 1 in the proof of Theorem 6 in [6].

Theorem C (Peng et al. [6]). *If $G_t^s(a,b;c,d)$ and $G_{t'}^{s'}(a',b';c',d')$ are chromatically equivalent and $s+t = s'+t'$, then $G_{t'}^{s'}(a',b';c',d') \in \mathcal{C}_r(a,b;c,d)$, where $r = s+t$.*

The next result gives the chromatic polynomial of $G_t^s(a,b;c,d)$. In [1], Chao and Zhao also determined the chromatic polynomial of this graph, but we shall use the computed chromatic polynomial of $G_t^s(a,b;c,d)$ in [6] to prove our main results.

Theorem D (Peng et al. [6]). *Let the order of $G_t^s(a,b;c,d)$ be n ($n = a+b+c+d+r-2$), and $x = 1-\lambda$. Then we have*

$$P(G_t^s(a,b;c,d)) = \frac{(-1)^n x}{(x-1)^2} \cdot Q(G_t^s(a,b;c,d)),$$

where

$$Q(G_i^s(a,b;\,c,d)) \;=\; (x^{n+1} - x^{a+b+r} - x^{c+d+r} + x^{r+1} - x) - (1+x+x^2) + (x+1)(x^a + x^b + x^c + x^d) - (x^{a+c} + x^{a+d} + x^{b+c} + x^{b+d}).$$

2. Main Theorem

In this section, we shall characterize the chromaticity of the family $C_r(a,b;\,c,d)$ when $\min\{a,b,c,d\} = r+2$, which gives us two counterexamples for the Conjecture.

Theorem 1. *The family of graphs in $C_r(a,b;\,c,d)$ is a chromatic equivalence class if $r \geq 2$ and $\min\{a,b,c,d\} = r+2$, except the two families $C_r(r+2,b;\,b+1,b+r+2)$ and $C_r(r+2,c+r+2;\,c,c+1)$.*

Proof. Let $G = G_i^s(a,b;\,c,d) \in C_r(a,b;\,c,d)$ and $H \sim G$. By Lemma 4 and Theorem 2 in [1], $H \cong G_{t'}^{s'}(a',b';\,c',d')$, where $a',b',c',d' \geq 1$. Let $r' = s' + t'$. If $r' = r$, then by Theorem C, $H \in C_r(a,b;\,c,d)$. Now assume $r' \neq r$. We solve the equation $Q(G) = Q(H)$. After cancelling the terms x^{n+1}, $-x$ and $-(1+x+x^2)$, we have $Q_1(G) = Q_1(H)$ where

$$Q_1(G) = x^{r+1} + (x+1)(x^a + x^b + x^c + x^d) - x^{r+a+b} - x^{r+c+d} - x^{a+c} - x^{a+d} - x^{b+c} - x^{b+d},$$

$$Q_1(H) = x^{r'+1} + (x+1)(x^{a'} + x^{b'} + x^{c'} + x^{d'}) - x^{r'+a'+b'} - x^{r'+c'+d'} - x^{a'+c'} - x^{a'+d'} - x^{b'+c'} - x^{b'+d'},$$

and $a+b+c+d+r = a'+b'+c'+d'+r'$; $\;a \leq b,\ a \leq c \leq d;\ a' \leq b',\ a' \leq c' \leq d'$.

Since by assumption $\min\{a,b,c,d\} = r+2$, the term x^{r+1} in $Q_1(G)$ cannot be cancelled. Hence x^{r+1} is in $Q_1(H)$ and this implies $r+1 = \min\{r'+1, a', b', c', d'\}$. Thus $r'+1 = r+1$ or $a' = r+1$. By our assumption, we must have $a' = r+1$. Since $a = \min\{a,b,c,d\} = r+2$, we have $Q_2(G) = Q_2(H)$ where

$$Q_2(G) = x^{r+3} + (x+1)(x^b + x^c + x^d) - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$

$$Q_2(H) = x^{r'+1} + (x+1)(x^{b'} + x^{c'} + x^{d'}) - x^{r'+r+b'+1} - x^{r'+c'+d'} - x^{r+c'+1} - x^{r+d'+1} - x^{b'+c'} - x^{b'+d'},$$

and $b+c+d+r+1 = b'+c'+d'+r'$;

$$r+2 = a \leq b,\ r+2 \leq c \leq d,\ r+1 \leq b',\ r+1 \leq c' \leq d'.$$

The lowest power positive term in $Q_2(G)$ cannot be cancelled, hence we have $\min\{b',c',d',r'+1\} \geq r+2$. The term x^{r+3} in $Q_2(G)$ cannot be cancelled. Hence x^{r+3} is a term in $Q_2(H)$, and thus we have $\underline{r'+1 = r+3}$ or $\underline{b' = r+3}$ or $\underline{c' = r+3}$ or $\underline{c'+1 = r+3}$ or $d' = r+3$ or $d'+1 = r+3$ or $b'+1 = r+3$. Since $b,c,d \geq r+2$, we have $g(H) = g(G) \geq 2r+4$. So $b' \geq r+3$ because $a' = r+1$. Thus $b' = r+2$ is impossible. Also $d' = r+3$ or $d' = r+2$ imply that $c' = r+2$ or $c' = r+3$. Therefore we need to consider only the first four cases (underline).

<u>*Case 1*</u> Suppose $r'+1 = r+3$ (or $r' = r+2$). Then we have $Q_3(G) = Q_3(H)$ where

$$Q_3(G) = (x+1)(x^b + x^c + x^d) - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$

$$Q_3(H) = (x+1)(x^{b'} + x^{c'} + x^{d'}) - x^{2r+b'+3} - x^{r+c'+d'+2} - x^{r+c'+1} - x^{r+d'+1} - x^{b'+c'} - x^{b'+d'};$$

and $b+c+d = b'+c'+d'+1$;

$$r+2 \leq b,\ r+2 \leq c \leq d,\ r+3 \leq b',\ r+2 \leq c' \leq d'.$$

It is easy to see that $\min\{b,c,d\} = \min\{b',c',d'\}$. We consider two subcases: $b \leq c$ and $b > c$.

<u>*Subcase 1.1*</u> Suppose $b \leq c$. Then we have $\min\{b,c,d\} = b$ and $g(G) = a+b$. Also we have $b = b'$ (if $b' \leq c'$) or $b = c'$ (if $b' > c'$). If $b = b'$, then $g(H) = a' + b' = g(G) = a+b$, and we have $a = a'$, a contradiction (since $a = r+2$ and $a' = r+1$). Hence we have $b = c'$ and $g(G) = a+b = a+c' = r+2+c'$.

Then $g(H)$ is equal to either $a'+b'$ or $c'+d'$ or $a'+r'+c' = 2r+3+c'$. Since $g(H) = g(G)$, the last possibility is impossible. We now look at the other two possibilities.

<u>*Subcase 1.1.1*</u> Suppose $g(H) = a'+b' = r+1+b'$ $(b = c')$. Then $g(G) = r+2+c' = r+1+b'$ and we have $b' = c'+1 = b+1$. Moreover, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = (x+1)(x^c + x^d) - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$

$$Q_4(H) = (x+1)(x^{b'} + x^{d'}) - x^{2r+b+4} - x^{r+b+d'+2} - x^{r+b+1} - x^{r+d'+1} - x^{2b+1} - x^{b+d'+1};$$

and $c+d = b'+d'+1$; $r+2 \leq b \leq c \leq d$, $r+3 \leq b' = c'+1$, $b = c' \leq d'$.

It can be seen that the term x^c in $Q_4(G)$ cannot be cancelled since if $c = 2r+b+2$, then at least one of the terms $x^{b'}$ and $x^{b'+1}$ is neither cancelled in $Q_4(H)$ nor is a term of $Q_4(G)$. Thus we must have x^c in $Q_4(H)$. So we have $c = b'$ or $c = d'$.

Subcase 1.1.1.1 Suppose $c = b'$ (or $c = b+1$). Then we have $d = d'+1$ and from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_5(G) = Q_5(H)$ where

$$Q_5(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2};$$
$$Q_5(H) = x^{d-1} - x^{2r+b+4} - x^{r+b+d+1} - x^{r+b+1} - x^{r+d},$$

The terms x^{d+1} and x^{d-1} must be cancelled in $Q_5(G)$ and $Q_5(H)$, respectively; otherwise, there is no solution. The term x^{d+1} can be cancelled with $-x^{2r+b+2}$ or $-x^{r+c+2}$, and the term x^{d-1} can be cancelled with $-x^{2r+b+4}$ or $-x^{r+b+1}$. If $d+1 = 2r+b+2$ then $d-1 = 2r+b$ and $Q(G) = Q(H)$ has no solution. If $d+1 = r+c+2$, then $d-1 = r+c = r+b+1$ and we have many solutions: $a = r+2, c = b+1, d = b+r+2$; and $a' = r+1, b' = b+1, c' = b, d' = b+r+1$, and $r' = r+2$. In other words, we have $G_r^0(r+2,b;\, b+1, b+r+2) \sim G_{r+2}^0(r+1, b+1;\, b, b+r+1)$, but $G_{r+2}^0(r+1,b+1;\, b,b+r+1) \notin C_r(r+2,b;\, b+1,b+r+2)$. Note that $G_r^0(r+2,b;\, b+1,b+r+2) \not\cong G_{r+2}^0(r+1,b+1;\, b,b+r+1)$ where $b \geq r+2$.

Subcase 1.1.1.2 Suppose $c = d'$. Recall that we also have $b = c' = b'-1$. Then $d = b'+1 = b+2$ and from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_6(G) = Q_6(H)$ where

$$Q_6(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$
$$Q_6(H) = x^{d-1} - x^{2r+b+4} - x^{r+b+c+2} - x^{r+b+1} - x^{r+c+1} - x^{2b+1} - x^{b+c+1};$$

Now $x^{d-1} = x^{b+1}$ in $Q_6(H)$ cannot be cancelled because $b \leq c$ but x^{b+1} is not a term in $Q_6(G)$. This is a contradiction.

Subcase 1.1.2 Suppose $g(H) = c'+d'$. Then $g(G) = r+2+c' = c'+d'$, thus $d' = r+2$. Since $r+2 \leq b = c' \leq d' = r+2$, we have $c' = r+2 = b$. From $Q_3(G) = Q_3(H)$, we have $Q_7(G) = Q_7(H)$ where

$$Q_7(G) = (x+1)(x^c + x^d) - x^{3r+4} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{r+2+c} - x^{r+2+d};$$
$$Q_7(H) = (x+1)(x^{b'} + x^{r+2}) - x^{2r+b'+3} - x^{3r+6} - 2x^{2r+3} - 2x^{b'+r+2} - x^{b'+r+2}.$$

and $c+d = b'+r+3$;

$$a = b = r+2 \leq c \leq d,\ a' = r+1,\ b' \geq r+3,\ c' = r+2 = d'.$$

The term x^{r+2} in $Q_7(H)$ cannot be cancelled, therefore x^{r+2} is a term of $Q_7(G)$ and $c = r+2$. Thus $d = b'+1$ and after cancelling equal terms, we have $Q_8(G) = Q_8(H)$ where

$$Q_8(G) = x^{d+1} - x^{3r+4} - x^{2r+2+d} - 2x^{2r+4} - 2x^{r+d+2};$$
$$Q_8(H) = x^{d-1} - x^{2r+d+2} - x^{3r+6} - 2x^{2r+3} - 2x^{d+r+1}.$$

Note that x^{d+1} and x^{d-1} must be cancelled in $Q_8(G)$ and $Q_8(H)$ respectively, but this is impossible. Therefore there is no solution.

Subcase 1.2 Suppose $b > c$. Thus $\min\{b,c,d\} = c$. So we have $c = b'$ (that is if $\min\{b',c',d'\} = b'$) or $c = c'$ (that is if $\min\{b',c',d'\} = c'$).

Subcase 1.2.1 Suppose $c = b'$. Then $g(H) = a'+b' = r+1+b' = g(G)$. If $g(G) = a+b = r+2+b$, then $r+2+b = r+1+b' = r+1+c$ or $c = b+1$. This contradicting our assumption that $b > c$. If $g(G) = c+d$, then $c+d = r+1+c$ but $d \geq r+2$, a contradiction. If $g(G) = a+c+r = 2r+c+2$, then $2r+c+2 = r+1+c$ and this is a contradiction. Therefore $c = b'$ is impossible.

Subcase 1.2.2 Suppose $c = c'$. Then from $Q_3(G) = Q_3(H)$ we have $Q_9(G) = Q_9(H)$ where

$$Q_9(G) = (x+1)(x^b + x^d) - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$
$$Q_9(H) = (x+1)(x^{b'} + x^{d'}) - x^{2r+b'+3} - x^{r+c+d'+2} - x^{r+c+1} - x^{r+d'+1} - x^{b'+c} - x^{b'+d'};$$

and $b+d = b'+d'+1$, $a = r+2 \leq b$, $c \leq d$, $c < b$;

$$a' = r+1 \leq b',\ c' \leq d',\ c' < b',\ c \leq d',\ c \leq b',\ b' \geq r+3.$$

Note that there is at least one positive term in $Q_9(G)$ that cannot be cancelled by a negative term. This can be seen as follows. For the case of $b < d$, at least one of the terms x^b or x^{b+1} cannot be cancelled. Also for the case of $b \geq d$, at least one of the terms x^d or x^{d+1} cannot be cancelled. Now consider the positive terms in $Q_9(G)$ and $Q_9(H)$. Since $Q_9(G) = Q_9(H)$, we have eight possibilities: $b = b'$, $b = b'+1$, $b = d'$, $b = d'+1$, $b+1 = b'$, $b+1 = d'$, $d+1 = b'$, or $d+1 = d'$.

Subcase 1.2.2.1 Suppose $b = b'$. Then $d = d'+1$; and from $Q_9(G) = Q_9(H)$, after cancelling equal terms, we have $Q_{10}(G) = Q_{10}(H)$ where

$$Q_{10}(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+d};$$
$$Q_{10}(H) = x^{d-1} - x^{2r+b+3} - x^{r+c+d+1} - x^{r+c+1} - x^{r+d} - x^{b+d-1}.$$

The term $-x^{r+c+1}$ is in $Q_{10}(H)$, but $-x^{r+c+1}$ is not in $Q_{10}(G)$ (since $c < b$ and $c \le d$). Therefore it must be cancelled by a positive term in $Q_{10}(H)$. Thus $r+c+1 = d-1$, hence $d+1 = r+c+3$ and the term x^{d+1} cannot be cancelled in $Q_{10}(G)$, which is a contradiction.

Subcase 1.2.2.2 Suppose $b = b' + 1$. Then $d = d'$, and from $Q_9(G) = Q_9(H)$, after cancelling equal terms, we have $Q_{11}(G) = Q_{11}(H)$ where

$$Q_{11}(G) = x^{b+1} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d},$$
$$Q_{11}(H) = x^{b-1} - x^{r+c+d+2} - x^{r+c+1} - x^{r+d+1} - x^{b+c-1} - x^{b+d-1}.$$

The term $-x^{r+c+1}$ is in $Q_{11}(H)$, but $-x^{r+c+1}$ is not in $Q_{11}(G)$ (since $c < b$ and $c \le d$). Therefore it must be cancelled by a positive term in $Q_{11}(H)$. Thus $r+c+1 = b-1$, hence $b+1 = r+c+3$ and the term x^{b+1} can be cancelled in $Q_{11}(G)$ if and only if $b+1 = r+c+3 = r+d+2$, or $d = c+1$. Thus, we have many solutions: $a = r+2$, $b = r+c+2$, $d = c+1$; $a' = r+1$, $b' = b-1 = r+c+1$, $c' = c$, $d' = d = c+1$, $r' = r+2$. In other words, we have $G^0_r(r+2, r+c+2\,;\, c, c+1) \sim G^0_{r+2}(r+1, r+c+1\,;\, c, c+1)$ but $G^0_{r+2}(r+1, r+c+1\,;\, c, c+1) \notin \mathcal{C}_r(r+2, r+c+2\,;\, c, c+1)$ for $c \ge r+2 \ge 4$. Note that $G^0_r(r+2, r+c+2\,;\, c, c+1) \not\cong G^0_{r+2}(r+1, r+c+1\,;\, c, c+1)$ for $c \ge r+2 \ge 4$.

Subcase 1.2.2.3 Suppose $b = d'$. Then $d = b'+1$, and from $Q_9(G) = Q_9(H)$, after cancelling equal terms, we have $Q_{12}(G) = Q_{12}(H)$ where

$$Q_{12}(G) = x^{d+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$
$$Q_{12}(H) = x^{d-1} - x^{2r+d+2} - x^{r+c+b+2} - x^{r+c+1} - x^{r+b+1} - x^{d+c-1} - x^{b+d-1}.$$

The term $-x^{r+c+1}$ is in $Q_{12}(H)$, but $-x^{r+c+1}$ is not in $Q_{12}(G)$ (since $c < b$ and $c \le d$). Therefore it must be cancelled by a positive term in $Q_{12}(H)$. Thus $r+c+1 = d-1$, hence $d+1 = r+c+3$ and the term x^{d+1} can be cancelled in $Q_{12}(G)$ if and only if $d+1 = r+c+3 = b+c$, or $b = r+3$. Since $r+2 \le c < b = r+3$, we have $c = r+2$, and hence $d = 2r+4$. So we have a solution for $Q(G) = Q(H)$: $a = r+2$, $b = r+3$, $c = r+2$, $d = 2r+4$; $a' = r+1$, $b' = d-1 = 2r+3$, $c' = c = r+2$, $d' = b = r+3$, $r' = r+2$. In other words, we have $G^0_r(r+2, r+3\,;\, r+2, 2r+4) \sim G^0_{r+2}(r+1, 2r+3\,;\, r+2, r+3)$ but $G^0_{r+2}(r+1, 2r+3\,;\, r+2, r+3) \notin \mathcal{C}_r(r+2, r+3\,;\, r+2, 2r+4)$ for $r \ge 2$. Note that $G^0_r(r+2, r+3\,;\, r+2, 2r+4) \not\cong G^0_{r+2}(r+1, 2r+3\,;\, r+2, r+3)$. This solution is a special case of solution in Subcase 1.2.2.2, where $c = r+2$.

Subcase 1.2.2.4 Suppose $b = d' + 1$. Then $d = b'$, and from $Q_9(G) = Q_9(H)$, after cancelling equal terms, we have $Q_{13}(G) = Q_{13}(H)$ where

$$Q_{13}(G) = x^{b+1} - x^{2r+b+2} - x^{r+c+d} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$
$$Q_{13}(H) = x^{b-1} - x^{2r+d+3} - x^{r+c+b+1} - x^{r+c+1} - x^{r+b} - x^{d+c} - x^{b+d-1}.$$

The term $-x^{r+c+1}$ is in $Q_{13}(H)$, but $-x^{r+c+1}$ is not in $Q_{13}(G)$ (since $c < b$ and $c \le d$). Therefore it must be cancelled by a positive term in $Q_{13}(H)$. Thus $r+c+1 = b-1$, hence $b+1 = r+c+3$ and the term x^{b+1} can be cancelled in $Q_{13}(G)$, only if $b+1 = r+c+3 = r+d+2$ or $d = c+1$. Thus the term $-x^{r+c+2}$ cannot be cancelled in $Q_{13}(G)$, and $-x^{r+c+2}$ is not a term of $Q_{13}(H)$, which is a contradiction.

Subcase 1.2.2.5 Suppose $b+1 = b'$. Then $d = d'+2$, and from $Q_9(G) = Q_9(H)$, after cancelling equal terms, we have $Q_{14}(G) = Q_{14}(H)$ where

$$Q_{14}(G) = x^b + (x+1)x^d - x^{2r+b+2} - x^{r+c+2} - x^{r+d+2} - x^{b+c} - x^{b+d};$$
$$Q_{14}(H) = x^{b+2} + (x+1)x^{d-2} - x^{2r+b+4} - x^{r+c+1} - x^{r+d-1} - x^{b+c+1} - x^{b+d-1}.$$

The term $-x^{r+c+1}$ is in $Q_{14}(H)$ but it is not in $Q_{14}(G)$ (since $c < b$ and $c \le d$). Hence $-x^{r+c+1}$ must be cancelled by a positive term in $Q_{14}(H)$. We have either $r+c+1 = b+2$ or $r+c+1 = d-2$ or $r+c+1 = d-1$. If $r+c+1 = b+2$, then the term $x^b = x^{r+c-1}$ cannot be cancelled in $Q_{14}(G)$ which means that x^b is a term of $Q_{14}(H)$. Thus we have either $b = d-2$ or $b = d-1$. In each case the term x^d cannot be cancelled in $Q_{14}(G)$, which is a contradiction.

If $r+c+1 = d-2$, then we have the term $x^{d-1} = x^{r+c+2}$ in $Q_{14}(H)$ and this term cannot be cancelled in $Q_{14}(H)$ because $r+2 \le c < b$. Since $-x^{r+c+2}$ occurs in $Q_{14}(G)$, we must have the term $2x^{r+c+2}$ in $Q_{14}(G)$, which is impossible.

If $r+c+1 = d-1$, then $d-2 = r+c$ and the term $x^{d-2} = x^{r+c}$ cannot be cancelled in $Q_{14}(H)$. Thus, we must have the term $x^{r+c} = x^{d-2}$ in $Q_{14}(G)$. This implies $b = r+c$. Now the term $x^{b+2} = x^{r+c+2}$ cannot be cancelled in $Q_{14}(H)$ because $r+2 \le c < b$. Since $-x^{r+c+2}$ occurs in $Q_{14}(G)$, we must have the term $2x^{r+c+2}$ in $Q_{14}(G)$, which is impossible. Therefore there is no solution.

In the remaining three possibilities, that is $b+1 = d'$, $d+1 = b'$, and $d+1 = d'$, there is no solution for $Q(G) = Q(H)$. The proof is similar to that of Subcase 1.2.2.5.

For Case 2 ($b' = r+3$), Case 3 ($c' = r+2$), and Case 4 ($c' = r+3$), there is no new solution for the equation $Q(G) = Q(H)$. The proof is similar to that of Case 1. The detailed proof can be obtained by e-mail from the second author or viewed at http://www.fsas.upm.edu.my/~yhpeng/publish/p2c234.pdf. □

Corollary. *We discover that the conjecture in [6] is only true for $r = 1$. For each $r \geq 2$, we provide two counterexamples as follows.*

- $G^0_r(r+2, b; b+1, b+r+2) \sim G^0_{r+2}(r+1, b+1; b, b+r+1)$ for $b \geq 4$ but $G^0_{r+2}(r+1, b+1; b, b+r+1) \notin \mathcal{C}_r(r+2, b; b+1, b+r+2)$.
- $G^0_r(r+2, c+r+2; c, c+1) \sim G^0_{r+2}(r+1, c+r+1; c, c+1)$ for $c \geq 4$ but $G^0_{r+2}(r+1, c+r+1; c, c+1) \notin \mathcal{C}_r(r+2, c+r+2; c, c+1)$.

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A few identities involving partitions with a fixed number of parts

Jean-Lou De Carufel

1 Notation and elementary identities

The partition function $P(n)$ gives the number of ways of writing an integer n as a sum of positive integers without regard to the order. Let $p(n,k)$ be the number of solutions of the diophantine equation

$$x_1 + x_2 + ... + x_k = n, \tag{1}$$

where $0 < x_1 \leq x_2 \leq ... \leq x_k$ and denote by $p_r(n,k)$ the number of solutions of (1) with $r \leq x_1 \leq x_2 \leq ... \leq x_k$.

By solving the recurrence relationship

$$p(n,m) - p(n-m,m) = p(n-1,m-1)$$

for small values of m, Colman [1] obtained formulas for $p(n,1)$, $p(n,2)$, $p(n,3)$, $p(n,4)$, $p(n,5)$ and $p(n,6)$. With simple combinatorial considerations together with the formulas

$$\left[\frac{n-1}{k}\right] + \left[\frac{n-2}{k}\right] + ... + \left[\frac{n-k}{k}\right] = n-k, \tag{2}$$

$$\sum_{i=1}^{3} \left\| \frac{(n-i)^2}{12} \right\| = \begin{cases} \left(\frac{n-2}{2}\right)^2 & \text{if } n \text{ is even,} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd,} \end{cases} \tag{3}$$

where $\|x\| = [x + 1/2]$ is the integer closest to x (and $[\,]$ is the greatest integer function), and the formulas for the sum of the k^{th} powers of the first n integers, we will derive some slick expressions for $p(n,1)$, $p(n,2)$, $p(n,3)$ and $p(n,4)$. Finally, we are going to use these identities together with a technique of Hirschhorn (see [2]) to give a formula for $p(n,5)$. Our approach and our method appear to be elementary and original and lead to formulas equivalent to those of Colman [1].

First, let us remark that

$$p(n,1) = 1, \quad p(n,2) = \left[\frac{n}{2}\right], \quad p_r(n,2) = \left[\frac{n}{2}\right] - (r-1)$$

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