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Clique-Coloring of $K_{3,3}$-Minor Free Graphs

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Abstract

A clique-coloring of a given graph G is a coloring of the vertices of G such that no maximal clique of size at least two is monocolored. The clique-chromatic number of G is the least number of colors for which G admits a clique-coloring. It has been proved that every planar graph is 3-clique colorable and every claw-free planar graph, different from an odd cycle, is 2-clique colorable. In this paper, we generalize these results to $K_{3,3}$-minor free ($K_{3,3}$-subdivision free) graphs.

Keywords Clique-coloring · Clique chromatic number · $K_{3,3}$-Minor free graphs · Claw-free graphs

Mathematics Subject Classification 05C15 · 05C10

1 Introduction

Graphs considered in this paper are all simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is called the order of G. The set of vertices adjacent to a vertex v is denoted by $N_G(v)$, and the size of $N_G(v)$ is called the degree of v and is denoted by $d_G(v)$. A vertex with degree zero is called an isolated vertex. The maximum degree of G is denoted by $\Delta(G)$. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. An independent set is a set of vertices in graph that does not induce any edge and the size of maximum independent set in G is written by $\alpha(G)$.

As usual, the complete bipartite graph with parts of cardinality m and n ($m, n \in \mathbf{N}$) is indicated by $K_{m,n}$. The graph $K_{1,3}$ is called a claw. The complete graph with n vertices

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$\{v_1, \ldots, v_n\}$ is denoted by K_n or $[v_1, \ldots, v_n]$. The graph $\bar{G}$ is the complement of G with the same vertex set as G, and uv is an edge in $\bar{G}$ if and only if it is not an edge in G. The path and the cycle of order n are denoted by P_n and C_n, respectively. The length of a path and a cycle is the number of its edges. A path with end vertices u and v is denoted by (u, v)-path.

Edge e is called an edge cut in connected graph G if $G/\{e\}$ is disconnected. A block in G is a maximal 2-connected subgraph of G. A chord of a cycle C is an edge not in C whose end vertices lie in C. A hole is a chordless cycle of length greater than three. A hole is said to be odd if its length is odd; otherwise, it is said to be even. Given a graph F, a graph G is called F-free if G does not contain any induced subgraph isomorphic with F. A graph G is a $(F_1, \ldots, F_k)$-free graph if it is F_i-free for all $i \in \{1, \ldots, k\}$. A graph G is claw-free (resp. triangle-free) if it does not contain $K_{1,3}$ (resp. K_3) as an induced subgraph.

By a subdivision of an edge $e = uv$, we mean replacing the edge e with a (u, v)-path. Any graph derived from graph F by a sequence of subdivisions is called a subdivision of F or an F-subdivision. The contraction of an edge e with endpoints u and v is the replacement of u and v with a vertex such that edges incident to the new vertex are the edges that were incident with either u or v except e; the obtained graph is denoted by $G \cdot e$. Graph F is called a minor of G (G is called F-minor graph) if F can be obtained from G by a sequence of vertex and edge deletions and edge contractions. Given a graph F, graph G is F-minor free if F is not a minor of G. Obviously, any graph G which contains an F-subdivision also has an F-minor. Thus an F-minor free graph is necessarily F-subdivision free, although in general the converse is not true. However, if F is a graph of the maximum degree at most three, any graph which has an F-minor also contains an F-subdivision. Thus, a graph is $K_{3,3}$-minor free if and only if it is $K_{3,3}$-subdivision free. By the well-known Kuratowski's theorem a graph is planar if and only if it is K_5-minor free and $K_{3,3}$-minor free. For further information on graph theory concepts and terminology we refer the reader to [17].

A vertex k-coloring of G is a function $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that for every two adjacent vertices u and v, $c(u) \neq c(v)$. The minimum integer k for which G has a vertex k-coloring is called the chromatic number of G and is denoted by $\chi(G)$. A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$, where V is the set of vertices of $\mathcal{H}$, and $\mathcal{E}$ is a family of non-empty subsets of V called hyperedges of $\mathcal{H}$. A *k-coloring* of $\mathcal{H} = (V, \mathcal{E})$ is a mapping $c : V \longrightarrow \{1, 2, \ldots, k\}$ such that for all $e \in \mathcal{E}$, where $|e| \geq 2$, there exist $u, v \in e$ with $c(u) \neq c(v)$. The *chromatic number* of $\mathcal{H}$, $\chi(\mathcal{H})$, is the smallest k for which $\mathcal{H}$ has a k-coloring. Indeed, every graph is a hypergraph in which every hyperedge is of size two and a k-coloring of such hypergraph is a usual vertex k-coloring.

A clique of G is a subset of mutually adjacent vertices of $V(G)$. A clique is said to be maximal if it is not properly contained in any other clique of G. We call *clique-hypergraph* of G, the hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ with the same vertices as G whose hyperedges are the maximal cliques of G of cardinality at least two. A k-coloring of $\mathcal{H}(G)$ is also called a *k-clique coloring* of G, and the chromatic number of $\mathcal{H}(G)$ is called the *clique-chromatic number* of G, and is denoted by $\chi_c(G)$. In other words, a k-clique coloring of G is a coloring of $V(G)$ such that no maximal clique in G is monochromatic, and $\chi_c(G) = \chi(\mathcal{H}(G))$. A clique coloring of $\mathcal{H}(G)$ is *strong* if no triangle of G is monochromatic. A graph G is *hereditary k-clique colorable* if G and all its induced subgraphs are k-clique colorable. The clique-hypergraph coloring problem was posed by Duffus et al. in [6]. To see more results on this concept, see [2,3,7,8,15].

Clearly, any vertex k-coloring of G is a k-clique coloring, whence $\chi_c(G) \leq \chi(G)$. It is shown that in general, clique coloring can be a very different problem from usual vertex coloring and $\chi_c(G)$ could be much smaller than $\chi(G)$ [2]. On the other hand, if G is triangle-free, then $\mathcal{H}(G) = G$, which implies $\chi_c(G) = \chi(G)$. Since the chromatic number of triangle-free graphs is known to be unbounded [10], we get that the same is true for the clique-chromatic number of triangle-free graphs. In addition, clique-chromatic number of claw-free graphs or even line graphs is not bounded. For instance for each constant k, there exists $N_k \in \mathbf{N}$ such that for each $n \geq N_k$, $\chi_c(L(K_n)) \geq k+1$ that $L(K_n)$ is line graph of complete graph K_n and is claw-free [2]. On the other hand, Défossez proved that a claw-free graph is hereditary 2-clique colorable if and only if it is odd-hole-free [5]. That is why recognizing the structure of graphs with bounded and unbounded clique-chromatic number could be an interesting problem.

For planar graphs, Mohar and Skrekovski in [9] proved the following theorem:

Theorem 1.1 [9] *Every planar graph is strongly* 3*-clique colorable.*

Moreover, Shan et al. in [12] proved the following theorem:

Theorem 1.2 [12] *Every claw-free planar graph, different from an odd cycle, is* 2*-clique colorable.*

Shan and Kang generalized the result of Theorem 1.1 to K_5-minor free graphs and the result of Theorem 1.2 to graphs which are claw-free and K_5-subdivision free [11] as follows:

Theorem 1.3 [11] *Every K_5-minor free graph is strongly* 3*-clique colorable.*

Theorem 1.4 [11] *Every graph which is claw-free and K_5-subdivision free, different from an odd cycle, is* 2*-clique colorable.*

In this paper, we generalize the result of Theorem 1.1 to $K_{3,3}$-minor free graphs and the result of Theorem 1.2 to claw-free and $K_{3,3}$-minor ($K_{3,3}$-subdivision) free graphs.

2 Preliminaries

In this section, we state the structure theorem of claw-free graphs that is proved by Chudnovsky and Seymour [4]. At first we need a number of definitions.

Two adjacent vertices u, v of graph G are called twins if they have the same neighbors in G, and if there are two such vertices, we say G admits twins. For a vertex v in G and a set $X \subseteq V(G) \backslash \{v\}$, we say that v is complete to X or X-complete if v is adjacent to every vertex in X; and that v is anticomplete to X or X-anticomplete if

v has no neighbor in X. For two disjoint subsets A and B of $V(G)$, we say that A is complete, respectively, anticomplete, to B, if every vertex in A is complete, respectively, anticomplete, to B. A vertex is called singular if the set of its non-neighbors induces a clique.

Let G be a graph and A, B be disjoint subsets of $V(G)$, the pair (A, B) is called homogeneous pair in G, if for every vertex $v \in V(G) \backslash (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. If one of the subsets A or B, for instance B is empty, then A is called a homogeneous set.

Let (A, B) be a homogeneous pair, such that A, B are both cliques, and A is neither complete nor anticomplete to B, and at least one of A, B has at least two members. In these conditions the pair (A, B) is called a W-join. A homogeneous pair (A, B) is non-dominating if some vertex of $V(G) \backslash (A \cup B)$ has no neighbor in $A \cup B$, and it is coherent if the set of all $(A \cup B)$-complete vertices in $V(G) \backslash (A \cup B)$ is a clique.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are non-empty and V_1 is anticomplete to V_2. The pair (V_1, V_2) is called a 0-join in G.

Next, suppose that V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

(1) A_i is a clique, and A_i, $V_i \backslash A_i$ are both non-empty;
(2) A_1 is complete to A_2;
(3) $V_1 \backslash A_1$ is anticomplete to V_2, and $V_2 \backslash A_2$ is anticomplete to V_1.

In these conditions, the pair (V_1, V_2) is a 1-join.

Now, suppose that V_0, V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following properties:

(1) A_i, B_i are cliques, $A_i \cap B_i = \emptyset$, and A_i, B_i and $V_i \backslash (A_i \cup B_i)$ are all non-empty;
(2) A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2;
(3) V_0 is a clique, and, for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \backslash (A_i \cup B_i)$.

The triple (V_0, V_1, V_2) is called a generalized 2-join, and, if $V_0 = \emptyset$, the pair (V_1, V_2) is called a 2-join.

The last decomposition is the following: Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$, there are cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

(1) the sets A_i, B_i, C_i are pairwise disjoint and have union V_i;
(2) V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2; and
(3) V_1, V_2 are both non-empty.

In these conditions it is said that G is a hex-join of V_1 and V_2.

Now we define classes $F_0, \ldots, F_7$ as follows:

- F_0 is the class of all line graphs.
- The icosahedron is the unique planar graph with 12 vertices of all degree five. For $k = 0, 1, 2, 3$, icosa(k) denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. The class F_1 is the family of all graphs G isomorphic to icosa(0), icosa(1), or icosa(2).

- Let H be the graph with vertex set $\{v_1, \ldots, v_{13}\}$, with the following adjacency: $v_1v_2 \ldots v_6v_1$ is a hole in G of length 6; v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$ and no other pairs are adjacent. The class F_2 is the family of all graphs G isomorphic to $H \backslash X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$.
- Let C be a circle, and $V(G)$ be a finite set of points of C. Take a set of subset of C homeomorphic to interval $[0, 1]$ such that there are not three intervals covering C and no two intervals share an end-point. Say that $u, v \in V(G)$ are adjacent in G if the set of points $\{u, v\}$ of C is a subset of one of the intervals. Such a graph is called circular interval graph. The class F_3 is the family of all circular interval graphs.
- Let H be the graph with seven vertices $h_0, \ldots, h_6$, in which $h_1, \ldots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let $H^{'}$ be the graph obtained from the line graph $L(H)$ by adding one new vertex, adjacent precisely to the members of $V(L(H)) = E(H)$ that are not incident with h_1 in H. Then $H^{'}$ is claw-free. Let F_4 be the class of all graphs isomorphic to induced subgraphs of $H^{'}$. Note that the vertices of $H^{'}$ corresponding to the members of $E(H)$ that are incident with h_1 in H form a clique in $H^{'}$. So the class F_4 is the family of graphs that is either a line graph or has a singular vertex.
- Let $n \geq 0$. Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. For $1 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j$, and let c_i be adjacent to a_j, b_j if and only if $i \neq j$. Let d_1, d_2, d_3, d_4, d_5 be five more vertices, where d_1 is $(A \cup B \cup C)$-complete; d_2 is complete to $A \cup B \cup \{d_1\}$; d_3 is complete to $A \cup \{d_2\}$; d_4 is complete to $B \cup \{d_2, d_3\}$; d_5 is adjacent to d_3, d_4; and there are no more edges. Let the graph just constructed be H. A graph $G \in F_5$ if (for some n) G is isomorphic to $H \backslash X$ for some $X \subseteq A \cup B \cup C$. Note that vertex d_1 is adjacent to all the vertices but the triangle formed by d_3, d_4 and d_5, so it is a singular vertex in G.
- Let $n \geq 0$. Let $A = \{a_0, \ldots, a_n\}$, $B = \{b_0, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. For $0 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j , b_j if and only if $i \neq j \neq 0$. Let the graph just constructed be H. A graph $G \in F_6$ if (for some n) G is isomorphic to $H \backslash X$ for some $X \subseteq (A \backslash \{a_0\}) \cup (B \backslash \{b_0\}) \cup C$.
- A graph G is prismatic, if for every triangle T of G, every vertex of G not in T has a unique neighbor in T. A graph G is antiprismatic if its complement is prismatic. The class F_7 is the family of all antiprismatic graphs.

The structure theorem in [4] is as follows:

Theorem 2.1 [4] If G is a claw-free graph, then either

- $G \in F_0 \cup \cdots \cup F_7$, or
- G admits either twins, a non-dominating W-join, a 0-join, a 1-join, a generalized 2-join, or a hex-join.

3 $K_{3,3}$-Minor Free Graphs

In this section, we focus on the clique chromatic number of $K_{3,3}$-minor free graphs. In particular, we prove that every $K_{3,3}$-minor free graph is strongly 3-clique colorable. Moreover, it is 2-clique colorable if it is claw-free and different from an odd cycle.

For this purpose, first we need the Wagner characterization of $K_{3,3}$-minor free graphs [14]. Let G_1 and G_2 be graphs with disjoint vertex-sets. Also, let $k \geq 0$ be an integer, and for $i = 1, 2$, let $X_i \subseteq V(G_i)$ be a set of cardinality k of pairwise adjacent vertices. For $i = 1, 2$, let G_i' be obtained from G_i by deleting a (possibly empty) set of edges with both ends in X_i. If $f : X_1 \longrightarrow X_2$ is a bijection, and G is the graph obtained from the union of G_1' and G_2' by identifying x with $f(x)$ for all $x \in X_1$, then we say that G is a k-sum of G_1 and G_2.

Theorem 3.1 [13,14] *A graph is $K_{3,3}$-minor free if and only if it can be obtained from planar graphs and complete graph K_5 by means of* 0-, 1-, 2-*sums.*

In order to make the above characterization easier, we use the structural sequence for $K_{3,3}$-minor free graphs. In fact, graph G is $K_{3,3}$-minor free if and only if there exists a sequence $\mathcal{T} = T_1, T_2, \ldots, T_r$, in which for each i, $1 \leq i \leq r$, T_i is either a planar graph or isomorphic with K_5, such that $G_1 = T_1$, and for each i, $2 \leq i \leq r$, G_i is obtained from disjoin union of G_{i-1} and T_i, or by gluing T_i to G_{i-1} on one vertex or one edge or two non-adjacent vertices and $G_r = G$. For a given $K_{3,3}$-minor free G, the sequence $\mathcal{T}$ is called a Wagner sequence.

Also we need following lemma proposed in [9]:

Lemma 3.2 [9] *Let G be a connected plane graph such that its outer cycle, C, is a triangle. If $\phi : V(C) \longrightarrow \{1, 2, 3\}$ is a clique coloring of induced subgraph C, then ϕ can be extended to a strong* 3*-clique coloring of G.*

In the following, we use the Wagner sequence to provide a strong 3-clique coloring for $K_{3,3}$-minor free graphs.

Theorem 3.3 *Every $K_{3,3}$-minor free graph is strongly* 3*-clique colorable.*

Proof Let G be a $K_{3,3}$-minor free graph. The assertion is trivial for $|V(G)| \leq 3$. So let $|V(G)| \geq 4$ and $\mathcal{T} = T_1, T_2, \ldots, T_r$ be a Wagner sequence of G. We use induction on r. If $r = 1$, then $G = T_1$ is either K_5 or a planar graph. If G is K_5, then the assertion is obvious, since by assigning color 1 to two vertices of K_5 and color 2 to two vertices of K_5 and color 3 to rest vertex, we have a strong 3-clique coloring of K_5. Also, if G is a planar graph, then the assertion follows directly from Theorem 1.1.

Now let $r \geq 2$. By the induction hypothesis G_{r-1} and T_r have strong 3-clique coloring. If G_r is 0-sum of G_{r-1} and T_r, then there is nothing to say. Suppose that G_r is obtained from G_{r-1} and T_r by gluing on vertex $\{v\}$. Thus, by a renaming of the colors, if it is necessary, we obtain a strong 3-clique coloring for G_r.

Next, we suppose that G_r is obtained from G_{r-1} and T_r by gluing on edge uv or two non-adjacent vertices u and v. If T_r is K_5, then we consider a strong 3-clique coloring of G_{r-1}, say ϕ, and extend it to a strong 3-clique coloring of G_r as follows: If $\phi(u) \neq \phi(v)$, then we assign three different colors $\{1, 2, 3\}$ to the other three vertices of K_5. If $\phi(u) = \phi(v)$, then we assign two different colors $\{1, 2, 3\}\backslash\{\phi(v)\}$ to the other three vertices of K_5. Obviously, the extended coloring is a strong 3-clique coloring of G_r.

Finally, let T_r be a planar graph. We consider a strong 3-clique coloring of G_{r-1}, say ϕ, and provide a strong 3-clique coloring of G_r as follows : If $\phi(u) \neq \phi(v)$ and $e = uv$ is a maximal clique of T_r, then suppose that $\phi^{'}$ is a strong 3-clique coloring of T_r. In this case, by a renaming the color of $\phi'(u)$ and $\phi'(v)$ in T_r, if it is necessary, we obtain a strong 3-clique coloring of G_r. If $e = uv$ is not a maximal clique in T_r, then there exists a triangle T containing e in T_r. Now we consider a planar embedding of T_r in which T is an outer face in it. Hence, by Lemma 3.2, it is enough to give a strong 3-clique coloring of outer cycle T of plane graph T_r. That is obviously possible by coloring the third vertex of T properly.

If $\phi(u) = \phi(v)$, then let $e = uv$ and $T_r^{'} = T_r \cdot e$. If there is no triangle consisting of $e = uv$ in $T_r^{'}$, then we consider a strong 3-clique coloring $\phi^{'}$ of plane graph $T_r^{'}$, such that $\phi^{'}(u) = \phi^{'}(v) = \phi(u) = \phi(v)$. Note that edge $e = uv$ is not maximal clique in G_{r-1}, so it is not maximal clique in G_r. Therefore, the coloring $\phi(x)$ for $x \in G_{r-1}$ and $\phi^{'}(x)$ for $x \in T_r \cdot e$ is a strong 3-clique coloring for G_r. If $e = uv$ is in triangle T in $T_r^{'}$, then we consider a planar embedding of $T_r^{'}$ in which T is an outer face in it. By Lemma 3.2, it is enough to give a 3-clique coloring of outer cycle T of plane graph $T_r^{'}$. Thus, we give $\phi'(u = v) = \phi(u) = \phi(v)$ and assign two different colors $\{1, 2, 3\}\backslash\{\phi(v)\}$ to other two vertices of T; then we extend ϕ' to a strong 3-clique coloring of $T_r^{'}$. This implies a strong 3-clique coloring of T_r as desired, and again we obtain a strong 3-clique coloring of G_r. □

The rest of this section deals with the proof that, every claw-free and $K_{3,3}$-minor free graph G, different from an odd cycle of order greater than three, is 2-clique colorable. For this purpose, we need two following theorems:

Theorem 3.4 [8] *If $G \in F_1 \cup F_2 \cup F_3 \cup F_5 \cup F_6$ or G admits a hex-join, different from an odd cycle of order greater than three, then G is 2-clique colorable.*

Theorem 3.5 [8] *Every connected claw-free graph G with maximum degree at most seven, not an odd cycle of order greater than three, is 2-clique colorable.*

From the proof of Theorem 3.5, we conclude the following corollary:

Corollary 3.6 *If G is a connected $K_{3,3}$-minor free graph which admits either twins, or a non-dominating W-join, or a coherent W-join, or a 1-join, or a generalized 2-join, except an odd cycle of order greater than three, then G is 2-clique colorable.*

According to Theorem 3.4 and Corollary 3.6, it is sufficient to show that every $K_{3,3}$-minor free graph $G \in F_0 \cup F_4 \cup F_7$ except an odd cycle of order greater than three, is 2-clique colorable. First we show this result for class F_0 (the class of line graphs).

Proposition 3.7 *Every $K_{3,3}$-minor free graph in F_0, different from an odd cycle of order greater than three, is 2-clique colorable.*

Proof Let G be a $K_{3,3}$-minor free line graph. The assertion is trivial for $|V(G)| \leq 3$. Now, let $|V(G)| \geq 4$. Let $\mathcal{T} = T_1, T_2, \ldots, T_r$ be a Wagner sequence of G. We use induction on r. If $r = 1$, then $G = T_1$ is either K_5 or a planar graph. If G is K_5, then the assertion is obvious. If G is a planar graph, then by Theorem 1.2, G has a 2-clique coloring, since every line graph is claw-free.

Now let $r \geq 2$. By the induction hypothesis G_{r-1} and T_r have 2-clique coloring. If G_r is 0-sum or 1-sum of G_{r-1} and T_r, then the result is obvious. Now, we suppose that G_r is 2-sum of G_{r-1} and T_r on edge uv. Note that if uv is an edge cut, then G can be considered as 1-sum of two graphs. So, later on we assume that uv is not an edge cut. If T_r is K_5 and ϕ is a 2-clique coloring of G_{r-1}, then we assign the colors $\phi(u)$ and $\phi(v)$ to vertices u, v in K_5 and give two different colors $\{1, 2\}$ to the other three vertices of K_5.

If T_r is a planar graph, then we have four possibilities:

(i) there exists 2-clique colorings ϕ and $\phi^{'}$ of G_{r-1} and T_r, such that $\phi(u) \neq \phi(v)$ and $\phi^{'}(u) \neq \phi^{'}(v)$;
(ii) there exists 2-clique colorings ϕ and $\phi^{'}$ of G_{r-1} and T_r, such that $\phi(u) = \phi(v)$ and $\phi^{'}(u) = \phi^{'}(v)$;
(iii) in every 2-clique colorings ϕ and $\phi^{'}$ of G_{r-1} and T_r, $\phi(u) \neq \phi(v)$ and $\phi^{'}(u) = \phi^{'}(v)$;
(iv) in every 2-clique colorings ϕ and $\phi^{'}$ of G_{r-1} and T_r, $\phi(u) = \phi(v)$ and $\phi^{'}(u) \neq \phi^{'}(v)$.

In the first two cases, only by a color renaming, if it is necessary, we obtain a 2-clique coloring for G_r. In the following, without loss of generality we consider the case (iii) and show that it is impossible:

The assumption (iii) concludes that vertex u (and v) in T_r belongs to a maximal clique C_u (and C_v) such that in every 2-clique coloring of T_r, $C_u \backslash \{u\}$ (and $C_v \backslash \{v\}$) is monochromatic. Hence, $u \notin C_v$ and $v \notin C_u$. This implies that, u has a non-neighbor vertex in C_v, say v', also v has a non-neighbor vertex in C_u, say u'. Moreover, assumption (iii) implies that uv is a maximal clique in G_{r-1}. Thus, there exist vertex $u'' \in N_{G_{r-1}}(u)$ that $u'' \notin N_{G_{r-1}}(v)$ (or $v'' \in N_{G_{r-1}}(v)$ that $v'' \notin N_{G_{r-1}}(u)$). Hence, edge uv among edges uu' and uu'' (or vv' and vv'') is a claw in G_r, that is a contradiction.

If in the operation 2-sum, the edge uv is deleted, then by the following argument, we could change the coloring of vertices in T_r such that $\phi^{'}(u) \neq \phi^{'}(v)$, that contradicts the assumption (iii). Note that since uv is not an edge cut in G_{r-1} and T_r, there are shortest (u, v)-paths $P : u_0 = uu_1 \ldots u_s = v$ in $T_r/\{uv\}$ and $Q : v_0 = vv_1 \ldots v_t = u$ in $G_{r-1}/\{uv\}$. Since G_r is claw-free, vertices u and v in T_r and G_{r-1} belong to only one maximal clique. If $d_{T_r}(u_i) = 2, i = 1, \ldots, s-1$ and $d_{G_{r-1}}(v_j) = 2, j = 1, \ldots, t-1$, then by (iii), the length of P is even and the length of Q is odd. This implies G_r is an odd cycle and contradicts our assumption. Thus, assume that $k \in \{0, 1, \ldots, s-1\}$ is the smallest indices that $d_{T_r}(u_k) \geq 3$ and $w \in N_{T_r}(u_k)$. Since G_r is claw free, we must have $w \in N_{T_r}(u_{k+1})$. Let C be a unique maximal clique consisting of $[u_k, u_{k+1}, w]$ (note that $N_{T_r}(u_k) \subseteq N_{T_r}(u_{k+1})$). If there exists a vertex in C that its color is $\phi^{'}(u_k)$, then we swap the colors of vertices on (u, u_k)-path in P. Thus, we will obtain a 2-

clique coloring of T_r such that u and v are assigned different colors. This contradicts the assumption (iii).

Now assume that the color of all vertices in C is different from $\phi^{'}(u_k)$. In this case, if there exists a vertex in C, say $w' \neq u_k$, such that C is a unique maximal clique contains w', then we assign $\phi^{'}(u_k)$ to w' and again swap the colors of vertices on (u, u_k)-path in P. Otherwise, every vertex in C belongs to a maximal clique other than C. In this case, if there exists a vertex $w' \in C$, such that $w' \in C'$, where C and C' are maximal cliques in different blocks of T_r, then we swap the color of vertices in the component of $T_r/\{w'\}$ consisting of C', assign $\phi^{'}(u_k)$ to w' and again swap the colors of vertices on (u, u_k)-path in P. Thus, we will obtain a 2-clique coloring of T_r such that u and v are assigned different colors. This contradicts the assumption (iii).

The remaining case is that all vertices in C belong to some other maximal cliques and all cliques are in one block in T_r. In this case, let l be the smallest indices that there exists a path from u_l to some vertices in $C/\{u_k, u_{k+1}\}$, whih we call (w, u_l)-path $P' : ww_1 \dots w_m = u_l$. Note that if there is no such a path, then we can consider graph G as a 2-sum of two graphs on edge $u_k u_{k+1}$, and we are done. If $m = 1$, then since P is a shortest path, we have $l = k + 2$. Therefore, the induced subgraph on vertices $\{u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}, w\}$ is one of the nine forbidden structures in line graphs (see [16]) (note that if $k = 0$ or $k = s - 2$, then vertex $u_{k-1} = v_{t-1}$ or $u_{k+3} = v_1$). Hence, $m \geq 2$. Also, w_{m-1} is adjacent to u_{l+1}, since T_r is claw free. Now, by considering the first internal vertices in P' and (u_{k+1}, u_l)-path in P with degree greater than two, we do the similar above discussion in order to change the color of vertices w or u_{k+1} and subsequently change the color of u. Therefore, if we could not do that, then we conclude that pattern of colors in these paths are $a, b, a, b \dots$, where $a, b \in \{1, 2\}$. Now, we have $\phi^{'}(w_{m-1}) = \phi^{'}(u_{l+1}) \neq \phi^{'}(u_l)$ or $\phi^{'}(w_{m-1}) \neq \phi^{'}(u_{l+1})$. In the former case, we swap the color of vertices in path $w_{m-1}w_{m-2} \dots w_1 w u_k \dots u$. In the latter case, we swap the color of vertices in path $u_l u_{l-1} \dots u_{k+1} u_k u_{k-1} \dots u$. Thus, in both cases, we obtain a 2-clique coloring for T_r such that the vertices u and v receive different colors and this contradicts the assumption (iii). Therefore, the cases (iii) and (iv) are impossible and the proof is complete. □

Now we show the 2-clique colorability of $K_{3,3}$-minor free graphs in class F_4. First, we need the following theorem:

Theorem 3.8 [2] *For any graph $G \neq C_5$ with $\alpha(G) \geq 2$, we have $\chi_c(G) \leq \alpha(G)$.*

Proposition 3.9 *Every $K_{3,3}$-minor free graph in F_4 is 2-clique colorable.*

Proof Let G be a graph in F_4. Since a graph in F_4 is a line graph or has a singular vertex, by Proposition 3.7 it is sufficient to consider graphs in F_4 with singular vertex. So by the constraction of graphs in F_4, we have $\alpha(G) \leq 3$. For case $\alpha(G) = 1$, the statement is obvious. If $\alpha(G) = 2$, then by Theorem 3.8, G is 2-clique colorable; otherwise, $\alpha(G) = 3$. Let x be a singular vertex and $S = \{r, s, t\}$ be a maximum independent set in G. Note that $x \notin S$, and since non-neighbor vertices of x induce a clique, vertices r, s are adjacent to x and t is not adjacent to x.

Now we propose a 2-clique coloring ϕ for G as follows: let $\phi(x) = 1$, $\phi(t) = 2$ and assign color 1 to every non-neighbor vertex of x except t. Now if x and t have more than one common neighbor, then assign color 2 to one of them and color 1 to the other vertices; otherwise, assign color 1 to their common neighbor. Finally, assign color 2 to the other adjacent vertices to x. It is easy to see that this assignment is a 2-clique coloring of G. □

Finally, we show the 2-clique colorability of $K_{3,3}$-minor free graphs in class F_7.

Proposition 3.10 *Every $K_{3,3}$-minor free graph in F_7 is 2-clique colorable.*

Proof Let G be a graph in F_7. Since G is an antiprismatic, $\bar{G}$ is prismatic. If $\bar{G}$ has no triangle, then $\alpha(G) = 2$, and by Theorem 3.8, is 2-clique colorable. Now let $T = [vuw]$ be a triangle in $\bar{G}$, and $S_1 = N_{\bar{G}}(v)\backslash\{u, w\}$, $S_2 = N_{\bar{G}}(u)\backslash\{v, w\}$ and $S_3 = N_{\bar{G}}(w)\backslash\{u, v\}$ be a partition of vertices $V(G) - \{v, u, w\}$.

Liang et al. in [8] prove that if

- (i) $|S_i| = 0$ for some $i = 1, 2, 3$, then G has a 2-clique coloring.
- (ii) $|S_i| = 1$ for some $i = 1, 2, 3$, then G has a 2-clique coloring.
- (iii) there is an edge xy in $\bar{G}$ such that for $i \neq j \in \{1, 2, 3\}$, x is an isolated vertex in $\bar{G}[S_i]$ and y is an isolated vertex in $\bar{G}[S_j]$, then there exists a 2-clique coloring of G.
- (iv) there exist $i \neq j \in \{1, 2, 3\}$ such that $S_i \cup S_j$ is an independent set in $\bar{G}$, then G has a 2-clique coloring.

In the following for the remaining cases, we provide a 2-clique coloring for G or we show that G is $K_{3,3}$-minor that is a contradiction. Let $S_1 = \{v_1, v_2\}$ and $S_2 = \{u_1, u_2\}$ and $S_3 = \{w_1, w_2\}$. There are $i \neq j, i, j \in \{1, 2, 3\}$, say $i = 1$, $j = 2$, such that v_1 is adjacent to v_2 in $\bar{G}$ and u_1 is adjacent to u_2 in $\bar{G}$; otherwise by (iii) or (iv), we have $\chi_c(G) \leq 2$. Hence, we have triangles $[uu_1u_2]$ and $[vv_1v_2]$ in $\bar{G}$. Since $\bar{G}$ is a prismatic v_1, v_2, w_1, w_2 have a unique neighbor in $[uu_1u_2]$ and u_1, u_2, w_1, w_2 have a unique neighbor in $[vv_1v_2]$. Thus, $\{u_1, u_2, v_1, v_2\}$ induces a cycle in $\bar{G}$ because, otherwise, for instance if u_1 and u_2 both are adjacent to v_1, then there exist two neighbors for u in triangle $[u_1u_2v_1]$. Without loss of generality, assume that u_1v_1 and u_2v_2 are edges in $\bar{G}$. That means, u_1v_2 and u_2v_1 are edges in G.

Now each two vertices w_1 and w_2 have unique neighbor in $[uu_1u_2]$ and $[vv_1v_2]$. If both vertices w_1 and w_2 are adjacent to u_1 (or u_2) and v_1 (or v_2) in $\bar{G}$, then there exists two neighbors for w_2 in triangle $[v_1u_1w_1]$ (or $[v_2u_2w_1]$) that contradicts $\bar{G}$ is prismatic. If vertices w_1 and w_2 are both adjacent to u_1 (or u_2) and v_2 (or v_1) in $\bar{G}$, then G has a $K_{3,3}$-minor, on vertices $\{w, w_1, w_2; u, v, v_1\}$ (or $\{w, w_1, w_2; u, v, v_2\}$). Note that if w_1 is adjacent to w_2 in $\bar{G}$, then we have triangle $[ww_1w_2]$ and since $\bar{G}$ is prismatic, vertices w_1 and w_2 cannot be both adjacent to one vertex of $\{v_1, v_2\}$ or $\{u_1, u_2\}$. If w_1 is adjacent to u_1 (or u_2) and v_1 (or v_2) and w_2 is adjacent to u_2 (or u_1) and v_2 (or v_2) in $\bar{G}$, then G has a $K_{3,3}$-minor, on vertices $\{w, w_1, w_2; u, v, v_2\}$ (or $\{w, w_1, w_2; u, v, v_1\}$). Hence, all cases above contradict that G is $K_{3,3}$-minor free or $\bar{G}$ is prismatic. Thus, it is enough to consider the two following remaining cases:

- w_1 is adjacent to u_1 and v_2, and w_2 is adjacent to u_2 and v_1 in $\bar{G}$ (Fig. 1b shows graph G).
- w_1 is adjacent to u_2 and v_1, and w_2 is adjacent to u_1 and v_2 in $\bar{G}$ (Fig. 1a shows graph G).

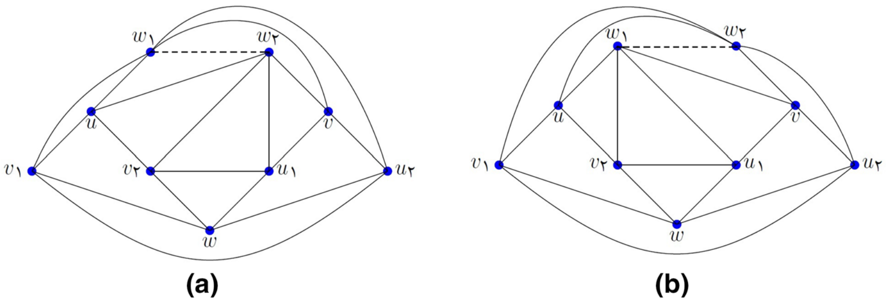

Fig. 1 Two $K_{3,3}$-minor free graphs

In both above cases G is a claw free planar graph and by Theorem 1.2 is 2-clique colorable (in Fig. 1, and the dashed lines show the edges that may exist or not exist in G).

Finally, let $|S_i| \geq 3$ for some $i = 1, 2, 3$, say $|S_1| \geq 2$, $|S_2| \geq 2$ and $S_3 = \{w_1, w_2, w_3\}$. Since such graphs contain the graphs with $|S_i| \leq 2$, $i = 1, 2, 3$ as subgraph, we only need to consider graphs that contains one of the two graphs shown in Fig. 1. By case (iv) there are $i \neq j \in \{1, 2, 3\}$ such that $\bar{G}[S_i]$ and $\bar{G}[S_j]$ both are not independent. Liang et al. in [8] show that $\bar{G}[S_i]$, $i \in \{1, 2, 3\}$, is not path and triangle. So we need to consider the case that $[uu_1u_2]$ and $[vv_1v_2]$ are triangles in $\bar{G}$, and $v_1w_3 \in E(\bar{G})$ or $v_2w_3 \in E(\bar{G})$. This implies G has a $K_{3,3}$-minor, on vertex set $\{w, w_1, w_2; u, v, v_2\}$ or $\{w, w_1, w_2; u, v, v_1\}$, respectively. Note that, when $[uu_1u_2]$ and $[ww_1w_2]$ are triangles in $\bar{G}$, the proof is similar. Therefore, when $|S_i| \geq 3$ for some $i = 1, 2, 3$, G is a $K_{3,3}$-minor, that is a contradiction. □

By Theorem 3.4, Corollary 3.6 and Propositions 3.7, 3.9, 3.10, the main result in this section is proved.

Theorem 3.11 *If G is claw-free and $K_{3,3}$-minor free graph except an odd cycle of order greater than three, then G is 2-clique colorable.*

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