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Local coloring of Kneser graphs☆

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Abstract

A local coloring of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where m_S is the number of edges of the induced subgraph $\langle S \rangle$. The maximum color assigned by a local coloring c to a vertex of G is called the value of c and is denoted by $\chi_\ell(c)$. The local chromatic number of G is $\chi_\ell(G) = \min\{\chi_\ell(c)\}$, where the minimum is taken over all local colorings c of G. The local coloring of graphs was introduced by Chartrand et al. [G. Chartrand, E. Salehi, P. Zhang, On local colorings of graphs, Congressus Numerantium 163 (2003) 207–221]. In this paper the local coloring of Kneser graphs is studied and the local chromatic number of the Kneser graph $K(n, k)$ for some values of n and k is determined.

Keywords: Local coloring; Local chromatic number; Kneser graphs

1. Introduction

A *standard coloring* or simply a (*vertex*) coloring of a graph G is a function $c : V(G) \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, having the property that $c(u) \neq c(v)$ for every pairs u, v of adjacent vertices of G. The *chromatic number* $\chi(G)$ is defined as the minimum number of colors used in any coloring of G. A *k-coloring* of G uses k colors. Define the *value* of a coloring c of G by $\chi(c) = \max\{c(v) : v \in V(G)\}$. Then $\chi(G) = \min\{\chi(c) : c \text{ is a coloring of } G\}$. In each k-coloring of G, the vertex set $V(G)$ is partitioned into nonempty subsets $V_1, V_2, \ldots, V_k$, where each set V_i, $1 \leq i \leq k$, is referred to as a *color class* with each vertex in V_i being assigned the color i, in fact each set V_i, $1 \leq i \leq k$, is an independent set.

Variations and generalizations of graph coloring have been studied by many authors and in many ways. The idea of defining the coloring of graphs by means of conditions placed on color classes was discussed in [5] and [6].

The standard definition of coloring can also be modified so that the local requirement that the adjacent vertices must be assigned distinct colors is replaced by a more global requirement.

For a graph G and a nonempty subset $S \subseteq V(G)$, let m_S denote the number of edges in the induced subgraph $\langle S \rangle$. A standard coloring of a graph G can be considered as a function $c : V(G) \to \mathbb{N}$ with the property that for every 2-element set $S = \{u, v\}$ of vertices of G, $|c(u) - c(v)| \geq m_S$.

Defining the standard coloring of a graph in this way is what suggested the extension of this concept was introduced in [1] and [2].

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Let G be a graph of order $n \geq 2$, and let k be a fixed integer with $2 \leq k \leq n$. A *k-local coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq k$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$. The maximum color assigned by a k-local coloring c to a vertex of G is called the *value* of c and is denoted by $\ell c_k(c)$. The *k-local chromatic number* of G is $\ell c_k(G) = \min\{\ell c_k(c)\}$, where the minimum is taken over all k-local colorings c of G. It follows that $\chi(G) = \ell c_2(G) \leq \ell c_3(G) \leq \cdots \leq \ell c_n(G)$.

The k-local coloring of graphs for $k = 3$ was discussed in [1,2] (also [10] and [11]). A 3-local coloring c of a graph G is also referred to as a *local coloring* of G and $\ell c_3(G)$ denoted by $\chi_\ell(G)$ which is also referred to as *local chromatic number* of G. It is written $\chi_\ell(c) = \ell c_3(c)$ and if $\chi_\ell(c) = \chi_\ell(G)$, then c is called a *minimum local coloring* of G. The whole of this paper considers the case $k = 3$. More specifically, a local coloring is a standard coloring with the additional requirement that any path of length 2 contains two vertices differing by ≥ 2 and any triangle contains two vertices differing by ≥ 3. The literature contains another (different) notion known as local chromatic number [3].

Therefore, the local chromatic number of G is slightly more global than the chromatic number of G since the conditions on colors that can be assigned to the vertices of G depend on subgraphs of order 2 and 3 in G rather than only on subgraphs of order 2.

Let $\{1, 2, \ldots, n\}$ be a set of size n, denoted by $[n]$. The Kneser graph with parameters n and k, $n \geq 2k$, denoted by $K(n, k)$ is a graph with k-element subsets of $[n]$ as its vertex set in which the two vertices are adjacent if and only if their corresponding k-element subsets are disjoint.

In this paper we study the local chromatic number of the Kneser graph $K(n, k)$. The chromatic number of the Kneser graph was settled in the famous work by Lovasz showing that it is $n - 2k + 2$. Here we conjecture and give some evidence that the local chromatic number of the Kneser graph is $2n - 4k + 2$. More specially, our main results show that the conjecture holds when $n = 2k$, $n = 2k + 1$ and $n = 2$. We also show that when n is large enough compared to k, one has that increasing n by 1 increases the local chromatic number by 2. This again agree the conjecture. It is also easy to see that our conjecture implies the Kneser conjecture, hence a proof is likely to be based on topological techniques.

2. Preliminaries

To prove our main results we need the following propositions.

Just as with standard coloring, where $\chi(H) \leq \chi(G)$ for any subgraph H of a graph G, it follows that $\chi_\ell(H) \leq \chi_\ell(G)$ as well.

For two graphs G and H, the graph $G \vee H$ is called the join of G and H, which is a graph given by $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Proposition A. *For every two graphs G and H, we have*

$$\chi_\ell(G \vee H) \leq \chi_\ell(G) + \chi_\ell(H) + 1.$$

Proof. Let c_1 and c_2 be local colorings of graphs G and H of value s_1 and s_2, respectively. We define a local coloring c of graph $G \vee H$ of value $s_1 + s_2 + 1$ as follows. For each vertex $v \in V(G \vee H)$, define

$$c(v) = \begin{cases} c_1(v) & v \in V(G), \\ s_1 + c_2(v) + 1 & v \in V(H). \end{cases}$$

It is easy to see that c is a local coloring of graph $G \vee H$ of value $s_1 + s_2 + 1$. Therefore, $\chi_\ell(G \vee H) \leq \chi_\ell(G) + \chi_\ell(H) + 1$. □

Let a standard k-coloring of a graph G be given, that is, the vertices of G have been assigned colors from $1, \ldots, k$ so that the adjacent vertices of G are colored differently. If we replace the color i by $2i - 1$ for every integer i, $1 \leq i \leq k$, then we obtain a local coloring for G. This gives the following proposition.

Proposition B ([1]). *For every graph G,*

$$\chi(G) \leq \chi_\ell(G) \leq 2\chi(G) - 1.$$

Proposition C (*[10]*). *If G is a nonbipartite graph, where the smallest degree of its vertices is at least* 3, *then* $\chi_\ell(G) \geq 4$.

Proof. Since G is not a bipartite graph, $\chi_\ell(G) \geq 3$. If $\chi_\ell(G) = 3$, then $\chi(G) = 3$. So for any local coloring c of G of value 3, there exists a vertex v such that $c(v) = 2$. The vertex v has at least three neighbors, at least two of them have colors either 1 or 3. Each case contradicts that c is a local coloring. Hence $\chi_\ell(G) \geq 4$. □

The following theorem is due to Hilton and Milner.

Theorem A (*[7]*). *If X is an independent set in the graph $K(n, k)$, $n \geq 2k$, and $|X| \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, then there exists $i \in [n]$ such that*

$$\bigcap_{A \in X} A = \{i\}.$$

Consider the graph $K(n, k)$ and set

$$X = \{A \in V(K(n, k)) : 1 \in A, A - \{1\} \subsetneqq \{k + 2, \ldots, n\}\} \cup \{\{2, \ldots, k + 1\}\}.$$

The set X is an independent set in $K(n, k)$ of size $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, but $\bigcap_{A \in X} A = \emptyset$.

3. Main results

In this section we find an upper bound for the local chromatic number of the Kneser graph $K(n, k)$ and determine the exact value of $\chi_\ell(K(n, k))$ for some values of n and k.

Proposition 1. *For every positive integer k,*

$$\chi_\ell(K(2k, k)) = 2.$$

Theorem 1. *For positive integers n and k, $n \geq 2k + 1$,*

$$\chi_\ell(K(n, k)) \leq \chi_\ell(K(n - 1, k)) + 2,$$
$$\chi_\ell(K(n, k)) \leq 2n - 4k + 2.$$

Proof. Let M be the set of all vertices that contain n. Hence M is an independent set in $K(n, k)$, and $K(n, k) \subseteq K(n - 1, k) \vee \langle M \rangle$. By Proposition A

$$\chi_\ell(K(n, k)) \leq \chi_\ell(K(n - 1, k)) + \chi_\ell(\langle M \rangle) + 1 = \chi_\ell(K(n - 1, k)) + 2.$$

Since $n \geq 2k + 1$, by continuing the above process, we have

$$\chi_\ell(K(n, k)) \leq \chi_\ell(K(2k, k)) + 2(n - 2k).$$

The second inequality follows from the first one by using Proposition 1. □

The following proposition shows that the upper bound in Theorem 1 is tight for $K(2k + 1, k)$.

Proposition 2. *For every positive integer k,*

$$\chi_\ell(K(2k + 1, k)) = 4.$$

Proof. If $k = 1$, then $K(2k + 1, k)$ is a complete graph of order 3, so $\chi_\ell(K(2k + 1, k)) = 4$. If $k \geq 2$, then $K(2k + 1, k)$ is a $(k + 1)$-regular graph which contains an odd cycle of size $2k + 1$. Therefore by Proposition C, $\chi_\ell(K(2k + 1, k)) \geq 4$. On the other hand by Theorem 1, $\chi_\ell(K(2k + 1, k)) \leq 4$. Hence

$$\chi_\ell(K(2k + 1, k)) = 4. \quad \square$$

Corollary 1. *For the Petersen graph $P = K(5, 2)$, $\chi_\ell(P) = 4$.*

Theorem 2. *If $K(n, k)$ has a minimum local coloring with a color class of size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, then*

$$\chi_\ell(K(n, k)) = \chi_\ell(K(n-1, k)) + 2.$$

Proof. Let c be a minimum local coloring of $K(n, k)$ and $X_a = c^{-1}(a) = \{A \in V(K(n, k)) : c(A) = a\}$, which satisfies $|X_a| \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$. By Theorem A, there exists $i \in [n]$ such that

$$\bigcap_{A \in X_a} A = \{i\}.$$

Claim. If there exists a vertex A such that $c(A) = a + 1$ or $c(A) = a - 1$, then $i \in A$.

Proof of Claim. Assume that $c(A) = a + 1$ or $c(A) = a - 1$. It is clear that A has at most one neighbor in X_a. Let $B = N(A) \cap X_a$ and $X' = (X_a - B) \cup \{A\}$. Since $|B| \leq 1$, $|X'| \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$ and X' is an independent set in $K(n, k)$. So by Theorem A, there is $j \in [n]$ such that

$$\bigcap_{C \in X'} C = \{j\}.$$

If $i \neq j$, then for each $C \in X' - A$, $\{i, j\} \subseteq C$ and we have the following inequality, which is impossible.

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2 \leq |X_a| \leq |X'| \leq \binom{n-2}{k-2} + 1.$$

Therefore $i = j$, and $i \in A$, as claimed. □

By the above claim the vertices with colors a, $a-1$ and $a+1$ induce an empty subgraph. Without loss of generality let $i = n$. Now we define coloring c' for $K(n-1, k)$. For each vertex $A \in V(K(n-1, k))$:

$$c'(A) = \begin{cases} c(A) & c(A) \leq a - 2, \\ c(A) - 2 & c(A) \geq a + 2. \end{cases}$$

Note that the vertices with colors $a-1$, a and $a+1$ contain n. It is obvious that c' is a local coloring of $K(n-1, k)$ and $\chi_\ell(c') \leq \chi_\ell(c) - 2$. Therefore,

$$\chi_\ell(K(n-1, k)) \leq \chi_\ell(K(n, k)) - 2.$$

Also by Theorem 1,

$$\chi_\ell(K(n-1, k)) \geq \chi_\ell(K(n, k)) - 2.$$

Hence,

$$\chi_\ell(K(n, k)) = \chi_\ell(K(n-1, k)) + 2. \quad \square$$

Theorem 3. *For every positive integer k, $k \geq 2$, if $n \geq 2k^2(k-1)$, then*

$$\chi_\ell(K(n, k)) = \chi_\ell(K(n-1, k)) + 2.$$

Proof. Assume by contradiction that $\chi_\ell(K(n, k)) < \chi_\ell(K(n-1, k)) + 2$. In particular this implies that $\chi_\ell(K(n, k)) \leq 2n - 4k + 1$. Therefore, there exist at most $2n - 4k + 1$ color classes. So by the pigeon-hole principle there is a color class of size at least $\left\lceil \frac{\binom{n}{k}}{2n-4k+1} \right\rceil$. In what follows we show that

$$\left\lceil \frac{\binom{n}{k}}{2n - 4k + 1} \right\rceil \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2,$$

which contradicts Theorem 2.

To see the inequality above, for $k = 2$, we have

$$\left\lceil \frac{\binom{n}{2}}{2n-7} \right\rceil \geq 4.$$

So it is sufficient to show that for $k \geq 3$

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2 \leq \frac{\binom{n}{k}}{2(n-2(k-1))}$$

or equivalently,

$$\left(1 - \frac{n-k-1}{n-1} \cdots \frac{n-2k+1}{n-k+1}\right) \frac{n-2(k-1)}{n} \leq \frac{1}{2k} - \frac{2(n-2(k-1))}{\binom{n-1}{k-1} n}.$$

For this purpose, we shall use the following inequalities [4]. For $3 \leq k \leq 4$, the inequality can be verified by straightforward calculations. Assume that $k \geq 5$. In the following, we shall use the fact that for any $x > -1$,

$$e^{\frac{x}{x+1}} \leq 1 + x \leq e^x.$$

By using the inequality above, for $i = 1, 2, \ldots, k-1$,

$$\frac{n-k-i}{n-i} \geq e^{-\frac{k}{n-k-i}}.$$

As $k \geq 5$ and $n \geq 2k^2(k-1)$, easy calculation shows that for $i = 1, 2, \ldots, k-1$,

$$\frac{1}{n-k-i} + \frac{1}{n-2k+i} \leq \frac{2}{n-2k+2} - \frac{4k(k-1)}{(n-k)^2(n-2k+2)}.$$

So,

$$\sum_{i=1}^{k-1} \frac{1}{n-k-i} \leq \frac{k-1}{n-2k+2} - \frac{2k(k-1)^2}{(n-k)^2(n-2k+2)}.$$

Hence,

$$\begin{aligned}\frac{n-k-1}{n-1} \cdots \frac{n-2k+1}{n-k+1} &\geq e^{-\sum_{i=1}^{k-1} \frac{k}{n-k-i}} \\ &\geq e^{-\left(\frac{k(k-1)}{n-2k+2} - \frac{2k^2(k-1)^2}{(n-k)^2(n-2k+2)}\right)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(1 - \frac{n-k-1}{n-1} \cdots \frac{n-2k+1}{n-k+1}\right) \frac{n-2(k-1)}{n} &\leq \left(1 - e^{-\left(\frac{k(k-1)}{n-2k+2} - \frac{2k^2(k-1)^2}{(n-k)^2(n-2k+2)}\right)}\right) \frac{n-2(k-1)}{n} \\ &\leq \left(\frac{k(k-1)}{n-2k+2} - \frac{2k^2(k-1)^2}{(n-k)^2(n-2k+2)}\right) \frac{n-2(k-1)}{n} \\ &= \frac{k(k-1)}{n} - \frac{2k^2(k-1)^2}{n(n-k)^2} \\ &\leq \frac{1}{2k} - \frac{2(n-2(k-1))}{\binom{n-1}{k-1} n}. \quad \square\end{aligned}$$

Lemma 1. *For* $n = 6, 7$, $\chi_\ell(K(n,2)) = 2n - 6$.

Proof. Let $n = 6$ and assume by contradiction that there exists a minimum local coloring c of $K(6, 2)$ of value 5. If c contains a color class of size at least 4, then by Theorem 2 and Proposition 2, $\chi_\ell(K(6, 2)) = \chi_\ell(K(5, 2)) + 2 = 6$, in contradiction. Therefore, in each minimum local coloring of $K(6, 2)$ each color class is of size at most 3. Let $X_i = c^{-1}(i)$, since $K(6, 2)$ has 15 vertices, $|X_i| = 3$, $1 \le i \le 5$. Each vertex in X_2 has exactly one neighbor in X_1, because no vertex in either X_1 or X_2 can have more than one edge going to the other set and if a vertex in X_2 has no neighbor in X_1, then we can find a local coloring with a color class of size 4. Therefore the induced subgraph $\langle X_1 \cup X_2\rangle$ is isomorphic to $3K_2$. If vertices A and B are adjacent vertices in $\langle X_1 \cup X_2\rangle$, then $|A \cup B| = 4$ and for each vertex $C \in X_1 \cup X_2 \setminus \{A, B\}$, we have $C \cap A \ne \emptyset$ and $C \cap B \ne \emptyset$. Moreover $|C| = 2$, so $C \subset A \cup B$. On the other hand $|X_1 \cup X_2| = 6$, so all of the 2-element subsets of $A \cup B$ belong to $X_1 \cup X_2$. Similarly for the color classes X_4 and X_5, if A' and B' are two adjacent vertices in $\langle X_4 \cup X_5\rangle$, then $|A' \cup B'| = 4$ and all of the 2-element subsets of $A' \cup B'$ belong to $X_4 \cup X_5$. Since $n = 6$, $|(A \cup B) \cap (A' \cup B')| \ge 2$. Therefore there is a common vertex in different color classes which is a contradiction. Therefore, $\chi_\ell(K(6, 2)) > 5$ and we are done.

Now let $n = 7$. Assume by contradiction that there exists a minimum local coloring c of $K(7, 2)$ of value 7, and let $X_i = c^{-1}(i)$. Similar to the above, we have $|X_i| = 3$, $1 \le i \le 7$, moreover if vertices A and B are adjacent vertices in $\langle X_1 \cup X_2\rangle$ and A' and B' are adjacent vertices in $\langle X_6 \cup X_7\rangle$, then $|A \cup B| = 4$ and $|A' \cup B'| = 4$. Also $X_1 \cup X_2$ and $X_6 \cup X_7$ contain all of the 2-element subsets of two 4-element sets, say $P = A \cup B$ and $Q = A' \cup B'$. Since $n = 7$, $|P \cap Q| \ge 1$. If $|P \cap Q| \ge 2$, then we must have a common vertex in different color classes, in contradiction, hence $|P \cap Q| = 1$, say $P = \{1, 2, 3, 4\}$ and $Q = \{1, 5, 6, 7\}$. Then the vertices $\{2, 5\}$, $\{3, 6\}$ and $\{4, 7\}$ induce a subgraph K_3 in color classes X_3, X_4 and X_5. This contradicts that c is a local coloring, hence $\chi_\ell(K(7, 2)) > 7$. □

Theorem 4. *For every positive integer n, $n \ge 4$,*

$$\chi_\ell(K(n, 2)) = 2n - 6.$$

Proof. By Theorem 1, $\chi_\ell(K(n, 2)) \le 2n - 6$. By Propositions 1 and 2 and Lemma 1, the statement is true for $n \le 7$. By Theorem 3, for $n \ge 8$, $\chi_\ell(K(n, 2)) = \chi_\ell(K(n - 1, 2)) + 2$. Therefore we are done. □

Propositions 1 and 2 and Theorem 4 show that in certain cases, the given upper bound in Theorem 1 is tight. We therefore propose the following conjecture.

Conjecture 1. *For every integers n and k, $n \ge 2k$,*

$$\chi_\ell(K(n, k)) = 2n - 4k + 2.$$

It was conjectured by Kneser in 1955 [8] and proved by Lovász in 1978 [9] that $\chi(K(n, k)) = n - 2k + 2$. By Proposition B, it is seen that $\chi(K(n, k)) \ge \frac{\chi_\ell(K(n,k))+1}{2}$. So the above conjecture generalizes the Kneser conjecture as well. Since our conjecture is stronger than the Kneser conjecture, a proof is likely to be based on topological techniques (either directly or indirectly).

Acknowledgments

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References

[1] G. Chartrand, E. Salehi, P. Zhang, On local colorings of graphs, Congressus Numerantium 163 (2003) 207–221.
[2] G. Chartrand, F. Saba, E. Salehi, P. Zhang, Local colorings of graphs, Utilitas Mathematica 67 (2005) 107–120.
[3] P. Erdös, Z. Füredi, A. Hanjal, P. Konjáth, V. Rödl, Á. Seress, Coloring graphs with locally few colors, Discrete Mathematics 59 (1987) 21–34.
[4] H. Hajiabolhassan, X. Zhu, Circular chromatic number of Kneser graphs, Journal of Combinatorial Theory Series B 88 (2003) 299–303.
[5] F. Harary, Conditional colorability in graphs. Graphs and Applications, Wiley, New York, 1985, pp. 127–136.
[6] F. Harary, K.F. Jones, Conditional colorability II: Bipartite variations, Congressus Numerantium 50 (1985) 205–218.
[7] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, The Quarterly Journal of Mathematics Oxford (2) 18 (1967) 369–384.
[8] M. Kneser, Aufgabe 300, Jahresber., Deutsch, Mathematics-Verein. 58 (1955) 27.
[9] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, Journal of Combinatorial Theory Series A 25 (1978) 319–324.
[10] B. Omoomi, A. Pourmiri, On the local colorings of graphs, Ars Combinatoria LXXXVI (January) (2008).
[11] B. Omoomi, A. Pourmiri, Locally rainbow graphs, Utilitas Mathematica (in press).