Locally Rainbow Graphs

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Abstract

A local coloring of a graph G is a function $c : V(G) \to N$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where m_S is the size of the induced subgraph $\langle S \rangle$. The maximum color assigned by a local coloring c to a vertex of G is called the value of c and is denoted by $\chi_\ell(c)$. The local chromatic number of G is $\chi_\ell(G) = \min\{\chi_\ell(c)\}$, where the minimum is taken over all local coloring c of G. If $\chi_\ell(c) = \chi_\ell(G)$, then c is called a minimum local coloring of G. A graph G is called locally rainbow if every minimum local coloring of G uses all of the colors $1, 2, \ldots, \chi_\ell(G)$. The concept of local coloring of graphs introduced by Chartrand et. al. in 2003. They suggested a conjecture on locally rainbow graphs. In this paper it is shown that their conjecture is true and for a given positive integer k, there exists a locally rainbow graph R_k with $\chi_\ell(R_k) = k$.

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for every positive integer n.

Remark. The proof of Theorem A not only shows $\chi_\ell(G) = 2k - 1$ for $G = K_{n_1,\ldots,n_k}$, where $k \geq 2$ and $n_i \geq 2$ for all $i \in \{1, 2, \ldots, k\}$, but that any minimum local coloring of G must color all the vertices in each partite set the same, namely, each of the colors $1, 3, \ldots, 2k-1$ is assigned to all vertices in a partite set.

It is well-known that if G is a graph with $\chi(G) = k$, then any coloring of G whose value is k must use all of the colors $1, 2, \ldots, k$. However if G is a graph with $\chi_\ell(G) = k$, then a minimum local coloring of G need not use all of colors $1, 2, \ldots, k$, although certainly the colors 1 and k must be used, as a simple example $\chi_\ell(K_3) = 4$.

For a graph G with $\chi_\ell(G) = k$, a minimum local coloring c of G is called a *local rainbow coloring* if for each integer i, $1 \leq i \leq k$, there is a vertex v of G for which $c(v) = i$, that is, c uses all of colors $1, 2, \ldots, k$. A graph G is called *locally rainbow* if every minimum local coloring of G is a local rainbow coloring.

In [1], for $1 \leq k \leq 5$, the locally rainbow graphs R_k are shown and the following conjecture is suggested.

Conjecture 1. *For every positive integer k, there exists a locally rainbow graph R_k with $\chi_\ell(R_k) = k$.*

In the following two theorems we prove that the conjecture above is true.

Theorem 1. *For every positive integer $k \geq 2$, there exists a locally rainbow graph R_{2k-1} with $\chi_\ell(R_{2k-1}) = 2k - 1$.*

Proof. To construct graph R_{2k-1}, first we consider the complete k-partite graph $G = K_{2,2,\ldots,2}$ and denote the parts of G by $V_1, \ldots, V_k$. By Theorem A, G has local chromatic number $2k - 1$ and in each

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1 Construction of Locally Rainbow Graphs

A *local coloring* of a graph G is a function $c : V(G) \longrightarrow N$ having the property that for each set $S \subseteq V(G)$ with $2 \le |S| \le 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \ge m_S$, where m_S is the size of the induced subgraph $\langle S \rangle$. The maximum color assigned by a local coloring c to a vertex of G is called the *value* of c and is denoted by $\chi_\ell(c)$. The *local chromatic number* of G is $\chi_\ell(G) = \min\{\chi_\ell(c)\}$, where the minimum is taken over all local coloring c of G. If $\chi_\ell(c) = \chi_\ell(G)$, then c is called a *minimum local coloring* of G. The local coloring of graphs introduced by Chartrand et. al. in [1] and [2].

Just as standard coloring, where $\chi(H) \le \chi(G)$ for any subgraph H of a graph G, it follows that $\chi_\ell(H) \le \chi_\ell(G)$ as well.

It is often useful to observe that if c is a local coloring of a graph G whose value is s, then the *complementary local coloring* $\bar{c}$ of c defined by $\bar{c}(v) = s + 1 - c(v)$ for all $v \in V(G)$ is a local coloring of G as well.

In [1] and [2] among other results the following result is established which we use to prove our main results.

Theorem A. *Let $G = K_{n_1,n_2,\ldots,n_{r+s}}$ be a complete multipartite graph, where r of the integers n_i are at least 2, the remaining s integers n_i are 1, and $r + s \ge 2$. Then*

$$\chi_\ell(G) = 2r + \left\lfloor \frac{3s-1}{2} \right\rfloor .$$

In particular,

$$\chi_\ell(K_n) = \left\lfloor \frac{3n-1}{2} \right\rfloor$$

minimum local coloring of G all the vertices in V_i have color $2i-1$ for $i = 1, 2, \ldots, k$. In the first step, we add $k^2 - k$ new vertices $\{u_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq k-1\}$ to $V(G)$ and then join each vertex u_{ij} to all vertices in $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k$.

In the second step, we add the complete graph K_{k-1} with vertex set $\{v_1, \ldots, v_{k-1}\}$ to the graph above, and then join the vertex v_j to the vertices u_{ij}, where $1 \leq i \leq k$ and $1 \leq j \leq k-1$. We denote this graph by R_{2k-1}.

Since each vertex u_{ij} has neighbors in the vertex set V_l, $1 \leq l \neq i \leq k$, the color of u_{ij} can not be $2l-1$. Moreover, if the color of u_{ij} is $2l$, $1 \leq l \leq k$, then we find an induced subgraph P_3 with colors $2l-1$ and $2l$. Therefore, it is seen that, in each minimum local coloring of the graph R_{2k-1} each vertex u_{ij}, $1 \leq j \leq k-1$, has color $2i-1$, for $i = 1, \ldots, k$. Hence the vertex v_i has color $2i$, for $i = 1, \ldots, k-1$. Therefore each minimum local coloring of graph R_{2k-1} uses all colors $1, 2, \ldots, 2k-1$, which means for every positive integer k, graph R_{2k-1} is a locally rainbow graph. □

In the following through some lemmas we prove that, for every positive integer k, there exists a locally rainbow graph R_{2k+2} with $\chi_\ell(R_{2k+2}) = 2k+2$.

Lemma 1. *Let $G = K_{n,1,1}$, where $n \geq 3$ and $V = \{v_1, \ldots, v_n\}, W = \{w\}$ and $Z = \{z\}$ be partite sets of G. In any minimum local coloring c of G the vertices in V have the same color. Moreover one of the following two possibilities exists; for each $v \in V$, $c(v) = 1$, $c(w) = 3$ and $c(z) = 4$ or for each $v \in V$, $c(v) = 4$, $c(w) = 1$ and $c(z) = 2$.*

Proof. By Theorem A, $\chi_\ell(G) = 4$. Since G has more than 4 vertices in any minimum local coloring c of G there are at least two vertices in V with the same color, say c_1. Without less of generality let $c(w) < c(z)$. We consider the following cases.

Case 1. $c_1 = 2$ or $c_1 = 3$.
Let $c_1 = 2$. Since vertices w, z and one vertex in V induced a subgraph K_3 and $\chi_\ell(K_3) = 4$, we must have $c(w) = 1$ and $c(z) = 4$. Now two vertices with color 2 in V with w induced a subgraph P_3 with the colors 1 and 2, which contradicts that c is a local coloring. The case $c_1 = 3$ is also failed by considering the complementary local coloring $\bar{c}$.

Case 2. $c_1 = 1$ or $c_1 = 4$.
Let $c_1 = 1$. Since vertices w, z and one vertex in V induced a subgraph K_3 and $\chi_\ell(K_3) = 4$, we must have $c(z) = 4$. Now two vertices with color 1 in V with w induced a subgraph P_3, therefore we must have $c(w) = 3$. For the case $c_1 = 4$, the other possibility follows by considering the complementary local coloring $\bar{c}$.

Now we show that the color of all vertices in partite set V are the same. To see this by contrary let $c_1 = 1$ and there exits a vertex, say u in V with color 2. So vertices u, w and z induced a subgraph K_3 with colors 2, 3 and 4, which contradicts that c is a local coloring. If $c_1 = 4$ then we have the same result by considering the complementary local coloring $\bar{c}$. □

Proposition 1. *Let $G_k = K_{n_1,\dots,n_k,1,1}$, where $n_i \geq 3$, $1 \leq i \leq k$, be a complete $(k+2)$-partite graph with partite sets $V_i = \{v^i_1, \dots, v^i_{n_i}\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$. In any minimum local coloring c of G_k the vertices in partite set V_i have the same color, say c_i and in the ordered set $c(V(G_k)) = \{c(v) \mid v \in V(G_k)\}$, the distance of every two consecutive colors is two, except $c(w)$ and $c(z)$, which $c(z) - c(w) = 1$. Moreover there are one of the three following possibilities. (Denote the $c(w)$ and $c(z)$ by c_w and c_z, respectively, and let $c_w < c_z$.)*

$$1 = c_1 < c_2 < \cdots < c_i < \cdots < c_k < c_w < c_z = 2k+2.$$

$$1 = c_w < c_z < c_1 < c_2 < \cdots < c_i < \cdots < c_k = 2k+2.$$

$$1 = c_1 < c_2 < \cdots < c_i < c_w < c_z < c_{i+1} < \cdots < c_k = 2k+2.$$

Proof. We prove the statement by induction on k. For $k = 1$, the statement is true by Lemma 1. Now let the statement be true for all $p < k$ and consider graph G_k which has at least $3k+2$ vertices. By Theorem A, $\chi_\ell(G_k) = 2k+2$, hence in each minimum local coloring c of G_k there are at least two vertices namely u and u' in partite set V_j with the same color, say a. Since u and u' with each vertex in the other partite sets in G_k induced a subgraph P_3, the color of each vertex $v \in V(G_k) - V_j$ is less than or equal to $a-2$ or greater than or equal to $a+2$. Now we consider the following cases.

Case 1. $a = 2k+2$ or $a = 1$.
Let $a = 2k+2$. Graph $G_k - V_j$ is a complete $(k+1)$-partite graph with $k-1$ partite sets of size at least three. In fact $G_k - V_j = G_{k-1}$ and the minimum local coloring c on $V(G_k) - V_j$ induced a minimum local coloring of G_{k-1} with value $\chi_\ell(G_{k-1}) = 2k$. Therefore by the induction hypothesis the color of all vertices in each partite sets are the same and one of the following possibilities appears.

$$1 = c_1 < c_2 < \cdots < c_i < \cdots < c_{k-1} < c_w < c_z = 2k.$$

$$1 = c_w < c_z < c_1 < c_2 < \cdots < c_i < \cdots < c_{k-1} = 2k.$$

$$1 = c_1 < c_2 < \cdots < c_i < c_w < c_z < c_{i+1} < \cdots < c_{k-1} = 2k.$$

Therefore by the induction hypothesis the distance of every two consecrative colors in above is two, except c_w and c_z. Moreover for each vertex $v \in V_j$, $c(v) = 2k+2$, because otherwise if there exits a vertex $v \in V_j$, such that $c(v) = a \neq 2k+2$, then we find an induced subgraph P_3 with colors $a-1$ and a or with colors a and $a+1$; or we have an induced complete graph K_3 with colors $a-2$, $a-1$ and a or with colors a, $a+1$ and $a+2$. Each of these cases contradicts that c is a local coloring. Therefore the statement is also true for graph G_k.

By considering the complementary local coloring $\bar{c}$, for the case $a = 1$ the result is obtained.

Case 2. $1 < a < 2k+2$.
In this case we define a local coloring c' of graph $G_{k-1} = G_k - V_j$. For each vertex $v \in V(G_{k-1})$, define

$$c'(v) = \begin{cases} c(v) & c(v) \leq a-2, \\ c(v)-2 & c(v) \geq a+2. \end{cases}$$

This coloring is a minimum local coloring of G_{k-1}, therefore by the induction hypothesis the statement is true for G_{k-1}. If b is the greatest color less than a to be used in local coloring c', then by adding 2 to the color of vertices with color greater than b in c' and use the same color as c for the vertices in V_j we get the local coloring c of G_k. Therefore the local coloring c has the desired properties because, for vertices v that $c'(v) \leq b$, we have $c(v) = c'(v)$ and for vertices v that $c'(v) > b$, we have $c(v) = c'(v) + 2$. Moreover the vertices in V_j all must have the same color, otherwise we find an induced subgraph in G_k with colors that contradicts the property of c. □

Consider the graph $G_k = K_{k+3,\cdots,k+3,1,1}$ with partite sets $V_i = \{v_1^i, \ldots, v_{k+3}^i\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$. Delete the edge set $\{v_i^s v_j^t \mid 4 \leq i,j \leq k+3, 1 \leq s \neq t \leq k\}$ in G_k. We called this new graph H_k and have the following lemma.

Lemma 2. *The graph H_k satisfies in Proposition 1 and $\chi_\ell(H_k) = 2k+2$.*

Proof. It is obvious that $G'_k = K_{3,\ldots,3,1,1}$, a complete $(k+2)$-partite graph with partite sets $V'_i = \{v_1^i, v_2^i, v_3^i\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$, is a subgraph of H_k. Also H_k is a subgraph of G_k. Therefore $\chi_\ell(H_k) = 2k+2$ and each minimum local coloring c of G_k is a minimum local coloring of G'_k. So G'_k satisfies in Proposition 1 and $c(v_1^i) = c(v_2^i) = c(v_3^i) = c_i$, $1 \leq i \leq k$. If there exists a vertex v_j^i

in H_k such that $c(v_j^i) = a \neq c_i$, then we find an induced subgraph P_3 with colors $a-1$ and a or with colors a and $a+1$; or we have an induced complete graph K_3 with colors $a-2$, $a-1$ and a or with colors a, $a+1$ and $a+2$. Each of these cases contradicts that c is a local coloring. Therefore for each vertex $v_j^i \in V_i$, $c(v_j^i) = c_i$ and one of the three possibilities in Proposition 1 appears. □

Theorem 2. *For each positive integer $k \geq 2$, there exists a locally rainbow graph R_{2k+2} with $\chi_\ell(R_{2k+2}) = 2k+2$.*

Proof. To construct graph R_{2k+2}, first we consider graph H_k constructed above and the complete graph K_k which $V(K_k) = \{u_1, \ldots, u_k\}$. We add the edges $E = \{u_i v_{3+i}^j |\ 1 \leq i \leq k, 1 \leq j \leq k\} \cup \{u_i w |\ 1 \leq i \leq k-1\} \cup \{u_k z\}$. We denote this new graph by R_{2k+2} and claim that $\chi_\ell(R_{2k+2}) = 2k+2$ and R_{2k+2} is a locally rainbow graph. We define a local coloring c of graph R_{2k+2} as follows. For each vertex $v \in V(R_{2k+2})$, define

$$c(v) = \begin{cases} 2i-1 & v \in V_i, 1 \leq i \leq k, \\ 2i & v = u_i \in V(K_k), 1 \leq i \leq k, \\ 2k+1 & v = w, \\ 2k+2 & v = z. \end{cases}$$

It is easy to see that c is a local coloring of R_{2k+2} with value $2k+2$.

Moreover each minimum local coloring of graph R_{2k+2} induced a minimum local coloring of graph H_k. Hence by Lemma 2 the colors of vertices in H_k have the properties of Proposition 1. By the construction above, it is obvious that the colors of vertices in $V(K_k)$ are different from the colors of partite sets $V_1, \ldots, V_k$. Also the colors of vertices in $V(K_k)$ in a local coloring c can not be the same as the colors $c(w)$ and $c(z)$, otherwise since $c(z) - c(w) = 1$, we find an induced subgraph P_3 with colors that contradicts the property of c. Therefore the colors of vertices in $V(K_k)$ are the rest of colors among the color set $\{1, 2, \ldots, 2k+2\}$. So R_{2k+2} is a locally rainbow graph as claimed. □

From Theorems 1 and 2 we conclude that, for every positive integer k, there exists a locally rainbow graph R_k with $\chi_\ell(R_k) = k$, which proves the Conjecture 1 is true.

References

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