Locally Rainbow Graphs

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Abstract

A local coloring of a graph $G$ is a function $c : V(G) \rightarrow \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where $m_S$ is the size of the induced subgraph $(S)$. The maximum color assigned by a local coloring $c$ to a vertex of $G$ is called the value of $c$ and is denoted by $\chi_{\ell}(c)$. The local chromatic number of $G$ is $\chi_{\ell}(G) = \min\{\chi_{\ell}(c)\}$, where the minimum is taken over all local coloring $c$ of $G$. If $\chi_{\ell}(c) = \chi_{\ell}(G)$, then $c$ is called a minimum local coloring of $G$. A graph $G$ is called locally rainbow if every minimum local coloring of $G$ uses all of the colors $1, 2, \ldots, \chi_{\ell}(G)$. The concept of local coloring of graphs introduced by Chartrand et. al. in 2003. They suggested a conjecture on locally rainbow graphs. In this paper it is shown that their conjecture is true and for a given positive integer $k$, there exists a locally rainbow graph $R_k$ with $\chi_{\ell}(R_k) = k$.

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**Remark.** The proof of Theorem A not only shows \( \chi_\ell(G) = 2k - 1 \)

for \( G = K_{n_1, \ldots, n_k} \), where \( k \geq 2 \) and \( n_i \geq 2 \) for all \( i \in \{1, 2, \ldots, k\} \),

but that any minimum local coloring of G must color all the vertices

in each partite set the same, namely, each of the colors 1, 3, \ldots, 2k - 1

is assigned to all vertices in a partite set.

It is well-known that if G is a graph with \( \chi(G) = k \), then any

coloring of G whose value is k must use all of the colors 1, 2, \ldots, k.

However if G is a graph with \( \chi_\ell(G) = k \), then a minimum local

coloring of G need not use all of colors 1, 2, \ldots, k, although certainly

the colors 1 and k must be used, as a simple example \( \chi_\ell(K_3) = 4 \).

For a graph G with \( \chi_\ell(G) = k \), a minimum local coloring c of

G is called a local rainbow coloring if for each integer i, 1 ≤ i ≤ k,

there is a vertex v of G for which c(v) = i, that is, c uses all of colors

1, 2, \ldots, k. A graph G is called locally rainbow if every minimum

local coloring of G is a local rainbow coloring.

In [1], for 1 ≤ k ≤ 5, the locally rainbow graphs \( R_k \) are shown

and the following conjecture is suggested.

**Conjecture 1.** For every positive integer k, there exists a locally

rainbow graph \( R_k \) with \( \chi_\ell(R_k) = k \).

In the following two theorems we prove that the conjecture above

is true.

**Theorem 1.** For every positive integer \( k \geq 2 \), there exists a locally

rainbow graph \( R_{2k-1} \) with \( \chi_\ell(R_{2k-1}) = 2k - 1 \).

**Proof.** To construct graph \( R_{2k-1} \), first we consider the complete

k-partite graph \( G = K_{2, 2, \ldots, 2} \) and denote the parts of G by \( V_1, \ldots, V_k \).

By Theorem A, G has local chromatic number 2k - 1 and in each
Key Words: local coloring, local chromatic number, locally rainbow graph.

1 Construction of Locally Rainbow Graphs

A local coloring of a graph $G$ is a function $c : V(G) \rightarrow N$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where $m_S$ is the size of the induced subgraph $(S)$. The maximum color assigned by a local coloring $c$ to a vertex of $G$ is called the value of $c$ and is denoted by $\chi_\ell(c)$. The local chromatic number of $G$ is $\chi_\ell(G) = \min\{\chi_\ell(c)\}$, where the minimum is taken over all local coloring $c$ of $G$. If $\chi_\ell(c) = \chi_\ell(G)$, then $c$ is called a minimum local coloring of $G$. The local coloring of graphs introduced by Chartrand et. al. in [1] and [2].

Just as standard coloring, where $\chi(H) \leq \chi(G)$ for any subgraph $H$ of a graph $G$, it follows that $\chi_\ell(H) \leq \chi_\ell(G)$ as well.

It is often useful to observe that if $c$ is a local coloring of a graph $G$ whose value is $s$, then the complementary local coloring $\bar{c}$ of $c$ defined by $\bar{c}(v) = s + 1 - c(v)$ for all $v \in V(G)$ is a local coloring of $G$ as well.

In [1] and [2] among other results the following result is established which we use to prove our main results.

Theorem A. Let $G = K_{n_1, n_2, \ldots, n_{r+s}}$ be a complete multipartite graph, where $r$ of the integers $n_i$ are at least 2, the remaining $s$ integers $n_i$ are 1, and $r + s \geq 2$. Then

$$\chi_\ell(G) = 2r + \left\lfloor \frac{3s - 1}{2} \right\rfloor.$$ 

In particular,

$$\chi_\ell(K_n) = \left\lfloor \frac{3n - 1}{2} \right\rfloor$$
minimum local coloring of $G$ all the vertices in $V_i$ have color $2i - 1$ for $i = 1, 2, \ldots, k$. In the first step, we add $k^2 - k$ new vertices \( \{u_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq k - 1\} \) to $V(G)$ and then join each vertex $u_{ij}$ to all vertices in $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k$.

In the second step, we add the complete graph $K_{k-1}$ with vertex set $\{v_1, \ldots, v_{k-1}\}$ to the graph above, and then join the vertex $v_j$ to the vertices $u_{ij}$, where $1 \leq i \leq k$ and $1 \leq j \leq k - 1$. We denote this graph by $R_{2k-1}$.

Since each vertex $u_{ij}$ has neighbors in the vertex set $V_l$, $1 \leq l \neq i \leq k$, the color of $u_{ij}$ can not be $2l - 1$. Moreover, if the color of $u_{ij}$ is $2l$, $1 \leq l \leq k$, then we find an induced subgraph $P_3$ with colors $2l - 1$ and $2l$. Therefore, it is seen that, in each minimum local coloring of the graph $R_{2k-1}$ each vertex $u_{ij}$, $1 \leq j \leq k - 1$, has color $2i - 1$, for $i = 1, \ldots, k$. Hence the vertex $v_i$ has color $2i$, for $i = 1, \ldots, k - 1$. Therefore each minimum local coloring of graph $R_{2k-1}$ uses all colors $1, 2, \ldots, 2k - 1$, which means for every positive integer $k$, graph $R_{2k-1}$ is a locally rainbow graph.

In the following through some lemmas we prove that, for every positive integer $k$, there exists a locally rainbow graph $R_{2k+2}$ with $\chi_\ell(R_{2k+2}) = 2k + 2$.

**Lemma 1.** Let $G = K_{n,1,1}$, where $n \geq 3$ and $V = \{v_1, \ldots, v_n\}, W = \{w\}$ and $Z = \{z\}$ be partite sets of $G$. In any minimum local coloring $c$ of $G$ the vertices in $V$ have the same color. Moreover one of the following two possibilities exists; for each $v \in V$, $c(v) = 1$, $c(w) = 3$ and $c(z) = 4$ or for each $v \in V$, $c(v) = 4$, $c(w) = 1$ and $c(z) = 2$.

**Proof.** By Theorem A, $\chi_\ell(G) = 4$. Since $G$ has more than 4 vertices in any minimum local coloring $c$ of $G$ there are at least two vertices in $V$ with the same color, say $c_1$. Without less of generality let $c(w) < c(z)$. We consider the following cases.
Case 1. $c_1 = 2$ or $c_1 = 3$.

Let $c_1 = 2$. Since vertices $w$, $z$ and one vertex in $V$ induced a subgraph $K_3$ and $\chi_\ell(K_3) = 4$, we must have $c(w) = 1$ and $c(z) = 4$. Now two vertices with color 2 in $V$ with $w$ induced a subgraph $P_3$ with the colors 1 and 2, which contradicts that $c$ is a local coloring. The case $c_1 = 3$ is also failed by considering the complementary local coloring $\bar{c}$.

Case 2. $c_1 = 1$ or $c_1 = 4$.

Let $c_1 = 1$. Since vertices $w$, $z$ and one vertex in $V$ induced a subgraph $K_3$ and $\chi_\ell(K_3) = 4$, we must have $c(z) = 4$. Now two vertices with color 1 in $V$ with $w$ induced a subgraph $P_3$, therefore we must have $c(w) = 3$. For the case $c_1 = 4$, the other possibility follows by considering the complementary local coloring $\bar{c}$.

Now we show that the color of all vertices in partite set $V$ are the same. To see this by contrary let $c_1 = 1$ and there exits a vertex, say $u$ in $V$ with color 2. So vertices $u$, $w$ and $z$ induced a subgraph $K_3$ with colors 2, 3 and 4, which contradicts that $c$ is a local coloring. If $c_1 = 4$ then we have the same result by considering the complementary local coloring $\bar{c}$. \hfill $\Box$

**Proposition 1.** Let $G_k = K_{n_1, \ldots, n_k, 1, 1}$, where $n_i \geq 3$, $1 \leq i \leq k$, be a complete $(k + 2)$-partite graph with partite sets $V_i = \{v_1^i, \ldots, v_{n_i}^i\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$. In any minimum local coloring $c$ of $G_k$ the vertices in partite set $V_i$ have the same color, say $c_i$ and in the ordered set $c(V(G_k)) = \{c(v) \mid v \in V(G_k)\}$, the distance of every two consecutive colors is two, except $c(w)$ and $c(z)$, which $c(z) - c(w) = 1$. Moreover there are one of the three following possibilities. (Denote the $c(w)$ and $c(z)$ by $c_w$ and $c_z$, respectively, and let $c_w < c_z$.)

1. $c_1 < c_2 < \cdots < c_i < \cdots < c_k < c_w < c_z = 2k + 2$.

2. $c_w < c_z < c_1 < c_2 < \cdots < c_i < \cdots < c_k = 2k + 2$.

3. $c_1 < c_2 < \cdots < c_i < c_w < c_z < c_{i+1} < \cdots < c_k = 2k + 2$. 

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Proof. We prove the statement by induction on \( k \). For \( k = 1 \), the statement is true by Lemma 1. Now let the statement be true for all \( p < k \) and consider graph \( G_k \) which has at least \( 3k + 2 \) vertices. By Theorem A, \( \chi^c(G_k) = 2k + 2 \), hence in each minimum local coloring \( c \) of \( G_k \) there are at least two vertices namely \( u \) and \( u' \) in partite set \( V_j \) with the same color, say \( a \). Since \( u \) and \( u' \) with each vertex in the other partite sets in \( G_k \) induced a subgraph \( P_3 \), the color of each vertex \( v \in V(G_k) - V_j \) is less than or equal to \( a - 2 \) or greater than or equal to \( a + 2 \). Now we consider the following cases.

Case 1. \( a = 2k + 2 \) or \( a = 1 \).

Let \( a = 2k + 2 \). Graph \( G_k - V_j \) is a complete \((k+1)\)-partite graph with \( k - 1 \) partite sets of size at least three. In fact \( G_k - V_j = G_{k-1} \) and the minimum local coloring \( c \) on \( V(G_k) - V_j \) induced a minimum local coloring of \( G_{k-1} \) with value \( \chi^c(G_{k-1}) = 2k \). Therefore by the induction hypothesis the color of all vertices in each partite sets are the same and one of the following possibilities appears.

\[
1 = c_1 < c_2 < \cdots < c_i < \cdots < c_{k-1} < c_w < c_z = 2k.
\]

\[
1 = c_w < c_z < c_1 < c_2 < \cdots < c_i < \cdots < c_{k-1} = 2k.
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\[
1 = c_1 < c_2 < \cdots < c_i < c_w < c_z < c_{i+1} < \cdots < c_{k-1} = 2k.
\]

Therefore by the induction hypothesis the distance of every two consecutive colors in above is two, except \( c_w \) and \( c_z \). Moreover for each vertex \( v \in V_j \), \( c(v) = 2k + 2 \), because otherwise if there exits a vertex \( v \in V_j \), such that \( c(v) = a \neq 2k + 2 \), then we find an induced subgraph \( P_3 \) with colors \( a - 1 \) and \( a \) or with colors \( a \) and \( a + 1 \); or we have an induced complete graph \( K_3 \) with colors \( a - 2, a - 1 \) and \( a \) or with colors \( a, a + 1 \) and \( a + 2 \). Each of these cases contradicts that \( c \) is a local coloring. Therefore the statement is also true for graph \( G_k \).

By considering the complementary local coloring \( \tilde{c} \), for the case \( a = 1 \) the result is obtained.
Case 2. $1 < a < 2k + 2$.

In this case we define a local coloring $c'$ of graph $G_{k-1} = G_k - V_j$. For each vertex $v \in V(G_{k-1})$, define
\[
c'(v) = \begin{cases} 
c(v) & c(v) \leq a - 2, \\
c(v) - 2 & c(v) \geq a + 2.
\end{cases}
\]

This coloring is a minimum local coloring of $G_{k-1}$, therefore by the induction hypothesis the statement is true for $G_{k-1}$. If $b$ is the greatest color less than $a$ to be used in local coloring $c'$, then by adding 2 to the color of vertices with color greater than $b$ in $c'$ and use the same color as $c$ for the vertices in $V_j$ we get the local coloring $c$ of $G_k$. Therefore the local coloring $c$ has the desired properties because, for vertices $v$ that $c'(v) \leq b$, we have $c(v) = c'(v)$ and for vertices $v$ that $c'(v) > b$, we have $c(v) = c'(v) + 2$. Moreover the vertices in $V_j$ all must have the same color, otherwise we find an induced subgraph in $G_k$ with colors that contradicts the property of $c$. $\square$

Consider the graph $G_k = K_{k+3, \ldots, k+3, 1, 1}$ with partite sets $V_i = \{v^i_1, \ldots, v^i_{k+3}\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$. Delete the edge set $\{v^i_s v^j_t \mid 4 \leq i, j \leq k + 3, 1 \leq s \neq t \leq k\}$ in $G_k$. We called this new graph $H_k$ and have the following lemma.

Lemma 2. The graph $H_k$ satisfies in Proposition 1 and $\chi(H_k) = 2k + 2$.

Proof. It is obvious that $G'_k = K_{3, \ldots, 3, 1, 1}$, a complete $(k+2)$-partite graph with partite sets $V'_i = \{v^i_1, v^i_2, v^i_3\}$, $1 \leq i \leq k$, $W = \{w\}$ and $Z = \{z\}$, is a subgraph of $H_k$. Also $H_k$ is a subgraph of $G_k$. Therefore $\chi(H_k) = 2k + 2$ and each minimum local coloring $c$ of $G_k$ is a minimum local coloring of $G'_k$. So $G'_k$ satisfies in Proposition 1 and $c(v^i_1) = c(v^i_2) = c(v^i_3) = c_i$, $1 \leq i \leq k$. If there exists a vertex $v^j_i$
in $H_k$ such that $c(v_j^i) = a \neq c_i$, then we find an induced subgraph $P_3$ with colors $a - 1$ and $a$ or with colors $a$ and $a + 1$; or we have an induced complete graph $K_3$ with colors $a - 2$, $a - 1$ and $a$ or with colors $a$, $a + 1$ and $a + 2$. Each of these cases contradicts that $c$ is a local coloring. Therefore for each vertex $v_j^i \in V_i$, $c(v_j^i) = c_i$ and one of the three possibilities in Proposition 1 appears. □

**Theorem 2.** For each positive integer $k \geq 2$, there exists a locally rainbow graph $R_{2k+2}$ with $\chi_f(R_{2k+2}) = 2k + 2$.

**Proof.** To construct graph $R_{2k+2}$, first we consider graph $H_k$ constructed above and the complete graph $K_k$ which $V(K_k) = \{u_1, \ldots, u_k\}$. We add the edges $E = \{u_jv_{j+i}^i| 1 \leq i \leq k, 1 \leq j \leq k\} \cup \{u_iw| 1 \leq i \leq k - 1\} \cup \{u_iz\}$. We denote this new graph by $R_{2k+2}$ and claim that $\chi_f(R_{2k+2}) = 2k + 2$ and $R_{2k+2}$ is a locally rainbow graph. We define a local coloring $c$ of graph $R_{2k+2}$ as follows. For each vertex $v \in V(R_{2k+2})$, define

$$c(v) = \begin{cases} 
2i - 1 & v \in V_i, 1 \leq i \leq k, \\
2i & v = u_i \in V(K_k), 1 \leq i \leq k, \\
2k + 1 & v = w, \\
2k + 2 & v = z. 
\end{cases}$$

It is easy to see that $c$ is a local coloring of $R_{2k+2}$ with value $2k + 2$.

Moreover each minimum local coloring of graph $R_{2k+2}$ induced a minimum local coloring of graph $H_k$. Hence by Lemma 2 the colors of vertices in $H_k$ have the properties of Proposition 1. By the construction above, it is obvious that the colors of vertices in $V(K_k)$ are different from the colors of partite sets $V_1, \ldots, V_k$. Also the colors of vertices in $V(K_k)$ in a local coloring $c$ can not be the same as the colors $c(w)$ and $c(z)$, otherwise since $c(z) - c(w) = 1$, we find an induced subgraph $P_3$ with colors that contradicts the property of $c$. Therefore the colors of vertices in $V(K_k)$ are the rest of colors among.
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the color set \{1, 2, \ldots, 2k + 2\}. So \(R_{2k+2}\) is a locally rainbow graph as claimed.  

From Theorems 1 and 2 we conclude that, for every positive integer \(k\), there exists a locally rainbow graph \(R_k\) with \(\chi_e(R_k) = k\), which proves the Conjecture 1 is true.

References
