Max-min total restrained domination number

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Abstract

Let G be a graph with vertex set V. A set $D \subseteq V$ is a total restrained dominating set of G if every vertex in V has a neighbor in D and every vertex in $V-D$ has a neighbor in $V-D$. The minimum cardinality of a total restrained dominating set of G is called the total restrained domination number of G, and is denoted by $\gamma_{tr}(G)$. Cyman and Raczek in 2006 showed that if G is a connected graph of order n and minimum degree δ such that $2 \leq \delta \leq n-2$, then $\gamma_{tr}(G) \leq n-\delta$. In this paper, we first introduce the concept of max-min total restrained domination number, denoted by $\gamma_{tr}^M(G)$, of G, and extend the above result by showing that $\gamma_{tr}(G) \leq \gamma_{tr}^M(G) \leq n-\delta$. We then proceed to establish that (1) $\gamma_{tr}(G) \leq n-2\delta$ if $n \geq 6$ and G contains a cut-vertex and (2) $\gamma_{tr}(G) \leq n-4$ if $n \geq 11$ and $\delta \geq 2$.

Key Words: Domination; Total restrained dominating set; Total restrained domination number; Max-min total restrained domination number.

1 Preliminaries

Let $G=(V,E)$ be a simple graph of order $n(G)$ and size $m(G)$. The *degree* of a vertex v in G is the number of vertices adjacent to v, and denoted by $deg_G(v)$. A vertex with no neighbor in G is called an *isolated vertex*. A vertex of degree one in G is called an *end vertex*, and the vertex adjacent to an end vertex is called a *support vertex*. The minimum degree and the maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If there is no confusion, we omit G in these notations. A graph $H = (V', E')$ is called a *subgraph* of G and denoted by $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. The *open neighborhood* of v is the set $N_G(v) := \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. For a set $X \subseteq V$, $N_G(X) = \bigcup_{v \in X} N_G(v)$ and $N_G[X] = \bigcup_{v \in X} N_G[v]$.

A set $D \subseteq V$ is a *dominating set* (DS) of G if every vertex in $V - D$ has a neighbor in D. The minimum cardinality of a dominating set of G is the *domination number* of G and denoted by $\gamma(G)$ (see [4, 5]). If, in addition, the induced subgraph $\langle D \rangle$ has no isolated vertex, then D is called a *total dominating set* (TDS) of G. The minimum cardinality of a TDS of G is called the *total domination number* and denoted by $\gamma_t(G)$. The notion of total domination in graphs was introduced by Cockayne et al. in [1] (see also [3, 4, 9]). Further, if D is a dominating set and the induced subgraph $\langle V - D \rangle$ has no isolated vertex, then D is called a *restrained dominating set* (RDS) of G. The minimum cardinality of a RDS of G is called the *restrained domination number* and denoted by $\gamma_r(G)$. The notion of restrained domination in graphs was introduced by Telle and Proskurowski implicitly in [10].

Throughout this paper, we assume that G contains no isolated vertex. A set $D \subseteq V$ is a *total restrained dominating set* of G (TRDS) if D is both a TDS and a RDS of G. Note that the set V is a TRDS of G. The minimum cardinality of a TRDS of G is called the *total restrained domination number* of G and denoted by $\gamma_{tr}(G)$. The concept of the total restrained domination was also introduced by Telle and Proskurowski implicitly in [10] and was formally presented in graph theory by De-Xiang Ma et al. in [8]. (See also [2, 6, 7].)

We now state some known results which are relevant to our work in this paper. For unexplained terms and symbols, see [11].

Proposition A. [2] *Every end vertex and support vertex in a graph G are in every TRDS of G.*

Proposition B. [8] *For path P_n and cycle C_n of order n,*

(i) $\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor, n \geq 2;$

(ii) $\gamma_{tr}(C_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor, n \geq 3.$

Theorem A. [2] *If G is a connected graph of order n and minimum degree*

δ such that $2 \leq \delta \leq n-2$, then

$$\gamma_{tr}(G) \leq n - \delta.$$

Theorem B. [6] *If G is a connected graph of order n, maximum degree Δ and minimum degree δ, where $2 \leq \delta \leq \Delta \leq n-2$, then*

$$\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1.$$

In Section 2 below, we shall introduce the concept of the max-min total restrained domination number of G, $\gamma_{tr}^M(G)$, and extend Theorem A by showing that $\gamma_{tr}(G) \leq \gamma_{tr}^M(G) \leq n-\delta$. In Section 3, we further show that (1) $\gamma_{tr}(G) \leq n - 2\delta$ if $n \geq 6$ and G contains a cut-vertex and (2) $\gamma_{tr}(G) \leq n-4$ if $n \geq 11$ and $\delta \geq 2$.

2 Max-min total restrained domination number

We begin with the introduction of the following notions.

Definition. *Let $G = (V, E)$ be a graph. For a vertex $v \in V$, define*

$$\mathrm{dom_{tr}}(v, G) := \min\{|S| \mid S \textit{ is a TRDS of } G \textit{ and } v \in S\}.$$

The **max-min total restrained domination number** *of G, denoted by $\gamma_{tr}^M(G)$, is defined by*

$$\gamma_{tr}^M(G) := \max\{\mathrm{dom_{tr}}(v, G) \mid v \in V\}.$$

Obviously, $\gamma_{tr}(G) = \min\{\mathrm{dom_{tr}}(v, G) \mid v \in V\}$. Thus, if G is a graph of order n, then

$$2 \leq \gamma_{tr}(G) \leq \gamma_{tr}^M(G) \leq n.$$

Remark 1. *The difference $\gamma_{tr}^M(G) - \gamma_{tr}(G)$ can be as large as desired. For example, if the graph G in Fig. 1 has k end vertices, then we have $\gamma_{tr}(G) = k+1$ and $\gamma_{tr}^M(G) = \mathrm{dom_{tr}}(v, G) = 2k+2$. On the other hand, for complete $t-$partite graph $G = K_{n_1, n_2, \ldots, n_t}$, $\gamma_{tr}^M(G) = \gamma_{tr}(G)$.*

It is not hard to see that if σ is an automorphism of graph G and D is a TRDS of G, then $D^\sigma = \{u^\sigma \mid u \in D\}$ is also a TRDS of G. So, if G is a vertex transitive graph and for two vertices u and v, $u^\sigma = v$, and S is a TRDS of G containing u, then S^σ is a TRDS of G containing v. We thus have the next proposition.

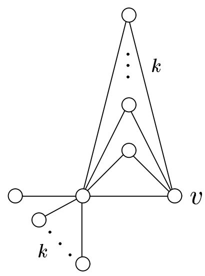

Figure 1: Graph G, where $\gamma_{tr}^{M}(G)-\gamma_{tr}(G)$ can be arbitrarily large.

Proposition 1. *For every vertex transitive graph G, $\gamma_{tr}^{M}(G)=\gamma_{tr}(G)$.*

Corollary 1. $\gamma_{tr}^{M}(C_n)=\gamma_{tr}(C_n)$, $\gamma_{tr}^{M}(K_n)=\gamma_{tr}(K_n)$.

Theorem 1. *For $n\geq 4$, $\gamma_{tr}^{M}(P_n)=\gamma_{tr}(P_{n-2})+2$.*

Proof. We prove the equality by induction on n. For $4\leq n\leq 5$, by Proposition B(i), $\gamma_{tr}(P_n)=n$. Since $n=\gamma_{tr}(P_n)\leq\gamma_{tr}^{M}(P_n)\leq n$, we have $\gamma_{tr}^{M}(P_n)=n=\gamma_{tr}(P_{n-2})+2$. Let $P_n=v_1v_2\dots v_n$ be a path of order n. Now assume that $n>5$ and the equality holds for every path P_m with $m<n$. Let D be an arbitrary TRDS of P_n containing v_4. By Proposition A, v_1 and v_2 belong to D; also since $\langle V-D\rangle$ has no isolated vertex, v_3 belongs to D. Thus, $D-\{v_1,v_2\}$ is a TRDS of $P_n-\{v_1,v_2\}$, and so $|D|\geq\gamma_{tr}(P_{n-2})+2$. Hence, $\gamma_{tr}^{M}(P_n)\geq\mathrm{dom}_{\mathrm{tr}}(v_4,P_n)\geq\gamma_{tr}(P_{n-2})+2$.

On the other hand, for $n\in\{6,7\}$, $\mathrm{dom}_{\mathrm{tr}}(v_4,P_n)=n$, and so $\gamma_{tr}^{M}(P_n)=n=\gamma_{tr}(P_{n-2})+2$. Now, if $n>7$, then for every $v\in V-\{v_1,v_2,v_3,v_4\}$, the union of every TRDS of $\langle V-\{v_1,v_2,v_3,v_4\}\rangle$ which contains v and $\{v_1,v_2\}$ is a TRDS of P_n containing v. Hence, $\mathrm{dom}_{\mathrm{tr}}(v,P_n)\leq\mathrm{dom}_{\mathrm{tr}}(v,P_{n-4})+2\leq\gamma_{tr}^{M}(P_{n-4})+2$. Therefore, by the induction hypothesis and Proposition B(i),

$$\mathrm{dom}_{\mathrm{tr}}(v,P_n)\leq\gamma_{tr}^{M}(P_{n-4})+2=(\gamma_{tr}(P_{n-6})+2)+2=\gamma_{tr}(P_{n-2})+2.$$

For the vertices v_3 and v_4 (similarly for v_{n-3} and v_{n-2}), $\mathrm{dom}_{\mathrm{tr}}(v_3,P_n)$ and $\mathrm{dom}_{\mathrm{tr}}(v_4,P_n)$ are at most $\gamma_{tr}(P_{n-2})+2$. Also, we have $\mathrm{dom}_{\mathrm{tr}}(v_1,P_n)=$

$\mathrm{dom}_{\mathrm{tr}}(v_2, P_n) = \gamma_{tr}(P_n) \le \gamma_{tr}(P_{n-2}) + 2$, in which the last inequality follows by Proposition B(i). Thus, for every $v \in V(P_n)$, $\mathrm{dom}_{\mathrm{tr}}(v, P_n) \le \gamma_{tr}(P_{n-2}) + 2$. Hence, $\gamma_{tr}^M(P_n) \le \gamma_{tr}(P_{n-2}) + 2$. Therefore, $\gamma_{tr}^M(P_n) = \gamma_{tr}(P_{n-2}) + 2$. ■

The next corollary follows readily from Theorem 1. Note that it provides graphs G which are not vertex transitive but $\gamma_{tr}(G) = \gamma_{tr}^M(G)$.

Corollary 2. *For $n \ge 2$ and $n \equiv 0$ or $1 \pmod 4$,*

$$\gamma_{tr}^M(P_n) = \gamma_{tr}(P_n).$$

Let v be a vertex in V and D be a subset of V. We define $\delta_v = \min\{deg_G(w) \mid w \in V - \{v\}\}$. Also, we call a vertex v a *bad vertex* with respect to D if it has no neighbor in D, or it is an isolated vertex in $\langle V - D\rangle$. Otherwise we call v a *good vertex* with respect to D. It is obvious that D is a TRDS of G if and only if G has no bad vertex with respect to D.

Theorem 2. *Let G be a connected graph of order n and minimum degree δ such that $2 \le \delta \le n-2$. Let $uv \in E$. Then there is a subset D of V with the following properties:*

1. *$v \in D$,*
2. *$V - D \subseteq N_G[u]$,*
3. *$|D| = n - \delta_v$,*
4. *$\langle V - D\rangle$ has a vertex of degree $\delta_v - 1$,*
5. *every vertex in $V - \{v\}$ is a good vertex with respect to D.*

Proof. Let $u_1, u_2, \ldots, u_{\delta_v - 1}$ be $\delta_v - 1$ neighbors of u other than v and $D := V - \{u, u_1, \ldots, u_{\delta_v-1}\}$. It is obvious that D has Properties 1 to 4. Moreover, the induced subgraph $\langle V - D\rangle$ has no isolated vertex. Also, $|V - D| = \delta_v$, hence, each vertex in $V - D$ has at least one neighbor in D. To see Property 5, we shall show that $\langle D\rangle$ has no isolated vertex, except possibly v or we find a set D' with the desired properties.

Suppose that $\langle D\rangle$ has an isolated vertex, say w, $w \ne v$. Then $N_G(w) \subseteq \{u, u_1, \ldots, u_{\delta_v-1}\}$ and $deg_G(w) \le \delta_v$. On the other hand, as $w \ne v$, $deg_G(w) \ge \delta_v$. Hence, $deg_G(w) = \delta_v$ and $N_G(w) = \{u_0(= u), u_1, \ldots, u_{\delta_v-1}\}$. Now we shall either find a subset D' of V having the desired properties or prove that $deg_G(u) = \delta_v$, which contradicts the existence of such a vertex w. For this purpose, assume that u_i is a neighbor of w with maximum degree among the neighbors of w. Let $D' := V - (N_G[w] - \{u_i\})$. It is obvious that $v \in D'$, $V - D' \subseteq N_G[u]$, $|D'| = n - \delta_v$ and the degree of vertex w in $\langle V - D' \rangle$ is equal to $\delta_v - 1$. Also, there is no isolated vertex in $\langle V - D' \rangle$ and every vertex in $V - D'$ has at least one neighbor in D'. Therefore, if there is no isolated vertex except possibly v in $\langle D' \rangle$, then D' is a desired subset.

Otherwise, $\langle D' \rangle$ has an isolated vertex other than v, say z. Then by the same reason for w, we get $deg_G(z) = \delta_v$ and $N_G(z) = \{w, u_0, u_1, u_2, \ldots, u_{\delta_v - 1}\} - \{u_i\}$. Since $N_G(w) = \{u_0, u_1, \ldots, u_{\delta_v - 1}\}$, the only neighbor of w in $V - D'$ is u_i. Hence, $z = u_i$; so $deg_G(u_i) = \delta_v$. By the choice of u_i, the degree of all neighbors of w is δ_v. Thus $deg_G(u_0) = \delta_v$, while the adjacency of w and $u_0 = u$ implies that $deg_G(u) > \delta_v$, a contradiction. Therefore, D or D' is a desired subset of V. ■

Corollary 3. *Let G be a connected graph of order n and minimum degree δ such that $2 \leq \delta \leq n - 2$. If $uv \in E$, such that $N_G[v] \not\subseteq N_G[u]$, then* $\mathrm{dom}_{\mathrm{tr}}(v, G) \leq n - \delta_v$.

Proof. Let D be a subset of V containing the vertex v with the properties as stated in Theorem 2. Since $N_G[v] \not\subseteq N_G[u]$ and $V - D \subseteq N_G[u]$, v is not an isolated vertex in D. Hence, G has no bad vertex with respect to D and D is a TRDS of G of size $n - \delta_v$ containing v. Clearly, $\mathrm{dom}_{\mathrm{tr}}(v, G) \leq n - \delta_v$. ■

Now we provide an upper bound for γ_{tr}^M of graphs.

Theorem 3. *If G is a connected graph of order n and minimum degree δ such that $2 \leq \delta \leq n - 2$, then*

$$\gamma_{tr}^M(G) \leq n - \delta.$$

Proof. Let v be an arbitrary vertex in G. We shall show that $\mathrm{dom}_{\mathrm{tr}}(v, G) \leq n - \delta$. If v has a neighbor, say u, such that $N_G[v] \not\subseteq N_G[u]$, then by Corollary 3, $\mathrm{dom}_{\mathrm{tr}}(v, G) \leq n - \delta_v \leq n - \delta$. If for every neighbor u of v, $N_G[u] = N_G[v]$, then by the connectedness of G, we conclude that G is a complete graph, which is a contradiction.

Now we assume that for each neighbor w of v, $N_G[v] \subseteq N_G[w]$ and v has a neighbor u such that $N_G[v] \subsetneqq N_G[u]$. Thus, $N_G[v]$ is a clique of order

$deg_G(v)+1$. If the degree of a neighbor of v other than u, say w, is more than $deg_G(v)$, then for graph $G'(=G-uw)$ we have $2 \leq \delta(G') \leq n-2$ and $N_{G'}[v] \nsubseteq N_{G'}[u]$. Hence, by Corollary 3, $\mathrm{dom_{tr}}(v,G') \leq n-\delta_v(G') \leq n-\delta(G')$. On the other hand, since $G' \subseteq G$, $\mathrm{dom_{tr}}(v,G) \leq \mathrm{dom_{tr}}(v,G')$; moreover, $deg_G(u) > deg_G(v) \geq \delta(G)$ and $deg_G(w) > deg_G(v) \geq \delta(G)$, which imply that $\delta(G') = \delta(G)$. Hence, $\mathrm{dom_{tr}}(v,G) \leq n-\delta(G)$. Thus, we assume that the degree of each neighbor of v other than u is equal to $deg_G(v)$.

Let z be a neighbor of u which is not in $N_G[v]$ and w be a neighbor of v other than u. We claim that $D := (V-(N_G(v)\cup\{z\}))\cup\{w\}$ is a TRDS of G. By the discussion above, if y is an isolated vertex in $\langle D\rangle$, then $deg_G(y)=2$ and $N_G(y)=\{z,u\}$. Hence, $\delta=2$. In this case, it can be easily seen that $V-\{y,z\}$ is a TRDS of G containing v of size $n-2$. So, suppose that $\langle D\rangle$ has no isolated vertex. Since $\delta \geq 2$ and u is the only neighbor of z in $V-D$, z has at least one neighbor in D. Also, every vertex in $V-D-\{z\}$ is adjacent to $v \in D$, thus, every vertex in $V-D$ has a neighbor in D. Moreover, since $V-D-\{z\} \subseteq N_G(v)$, the vertices in $V-D$ other than z induce a clique. Hence, every vertex in $V-D$ is adjacent to $u \in V-D$, and $\langle V-D\rangle$ has no isolated vertex. Therefore, D is a TRDS of G containing v. Hence, $\mathrm{dom_{tr}}(v,G) \leq n-deg_G(v) \leq n-\delta$, as required. ■

Remark 2. *The following examples show that the upper bound in Theorem 3 may no longer valid if any condition is violated.* (i) *If G is a star of order n, then $\gamma_{tr}^M(G) = n \nleq n-\delta$.* (ii) *If $\delta = n-1$, then $n-\delta = 1$; while for every graph G, $2 \leq \gamma_{tr}^M(G) \nleq n-\delta$.*

3 Upper bounds for γ_{tr}

In this section, we first give an upper bound for graphs with cut vertex, in terms of the order and the minimum degree of graph, and then characterize graphs of order n with total restrained domination number equal to $n-2$. Finally, we give an upper bound for the total restrained domination number of graphs in terms of order of graph.

The upper bound given in the next theorem improves the upper bound in Theorem A for graphs which are not 2-connected. We call a graph of order 5 formed by two triangles with a common vertex a *bow-tie*. In a connected graph contains a cut vertex, the blocks which contains exactly one cut vertex are called *end blocks*. It is obvious that such a graph has at least two end blocks.

Theorem 4. *If G is a connected graph of order n and minimum degree $\delta \geq 2$ containing a cut vertex and G is not a bow-tie, then*

$$\gamma_{tr}(G) \leq n - 2\delta.$$

Proof. Let B_1 and B_2 be two end blocks of G. Since $\delta(G) \geq 2$, $n(B_1) \geq 3$ and $n(B_2) \geq 3$; also, $\delta(B_1) \geq 2$ and $\delta(B_2) \geq 2$. Let v_1 and v_2 be the unique cut vertices in B_1 and B_2, respectively.

If $\delta(B_i) \leq n(B_i) - 2$, $i = 1, 2$, then by Theorem 2, $V(B_i)$ has a subset, D_i, such that $v_i \in D_i$, $|D_i| = n(B_i) - \delta_{v_i}(B_i) \leq n(B_i) - \delta$ and every vertex in $V(B_i) - \{v_i\}$ is a good vertex with respect to D_i. If $\delta(B_i) = n(B_i) - 1$, $i = 1, 2$, then let $D_i := \{v_i\}$. Thus, in each case, we have a subset D_i of $V(B_i)$ such that $v_i \in D_i$, $|D_i| \leq n(B_i) - \delta$ and every vertex in $V(B_i) - \{v_i\}$ is a good vertex with respect to D_i. Now let $D := D_1 \cup D_2 \cup (V(G) - (V(B_1) \cup V(B_2)))$, where $|D| \leq n - 2\delta$. If D is a TRDS of G, we are done.

If D is not a TRDS of G, then $G = B_1 \cup B_2$ and $v_1 = v_2$ and $|D| = |D_1| + |D_2| - 1 \leq n - 2\delta - 1$. If $D_1 = D_2 = \{v_1 (= v_2)\}$, since G is not a bow-tie, then at least one of B_1 or B_2 has more than 3 vertices. Suppose that $n(B_1) \geq 4$. In this case, since $\delta(B_1) = n(B_1) - 1$ and $\delta(B_2) = n(B_2) - 1$, the blocks B_1 and B_2 are complete subgraphs. Therefore, the set $D = \{v_1, u\}$, where $u \in B_1$, is a TRDS of G of order 2. It is obvious that $n(B_1) \leq \frac{n}{2}$ or $n(B_2) \leq \frac{n}{2}$. Thus, $\delta \leq \frac{n}{2} - 1$. Hence, $2 \leq n - 2\delta$ and D is a TRDS of order at most $n - 2\delta$.

Otherwise, without loss of generality suppose that $D_1 \neq \{v_1\}$. If $\delta_{v_1}(B_1) = 2$, then $\delta = 2$. In this case by Theorem 3, we have TRDS D'_1 and D'_2 of B_1 and B_2, respectively, containing $v_1 = v_2$, such that $|D'_1| \leq n(B_1) - 2$ and $|D'_2| \leq n(B_2) - 2$. Hence, $D'_1 \cup D'_2$ is a desired TRDS of G. Thus, assume that $\delta_{v_1}(B_1) > 2$. Since by the choice of D_1, D_1 has the properties of Theorem 2 and by assumption D is not a TRDS of G, v_1 is a bad vertex with respect to D_1. Let u be a vertex of degree $\delta_{v_1}(B_1) - 1$ in $V - D_1$, which exists by Theorem 2(4) and w be a neighbor of v_1 in $V - D_1$ other than u. It can be seen that $D \cup \{w\}$ is a TRDS of G of order $n - 2\delta$. ■

Let G be a graph of order n. In [2], those G with $\gamma_{tr}(G) = n$ were characterized. It is obvious that $\gamma_{tr}(G) \neq n - 1$. In the following theorem, we characterize the 2-connected graphs of order n with the total restrained domination number equal to $n - 2$. Obviously, for all 2-connected graphs G of order 4, $\gamma_{tr}(G) = n - 2$.

Theorem 5. *Let G be a 2-connected graph of order $n > 4$. Then $\gamma_{tr}(G) = n - 2$ if and only if $G = C_n$ for $n = 5, 6, 7$.*

Proof. If $G = C_n$, $5 \leq n \leq 7$, then by Proposition B(ii), $\gamma_{tr}(C_n) = n - 2\lfloor \frac{n}{4} \rfloor = n - 2$.
Now assume that $\gamma_{tr}(G) = n - 2$. By Theorem B, if $\Delta \leq n - 2$, then $\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1$. Hence, if $\Delta \geq 3$, then $\gamma_{tr}(G) \leq n - 3$. So, if $\gamma_{tr}(G) = n - 2$, then $\Delta = 2$ or $\Delta = n - 1$.

If $\Delta = 2$, then since G is 2-connected, G is a cycle. So by Proposition B(ii), $G = C_n$, where $5 \leq n \leq 7$.

If $\Delta = n-1$, then let v be a vertex of degree $n-1$. Since G is 2-connected, it has a vertex other than v, say u, with $deg_G(u) \geq 3$. Let w and z be the neighbors of u other than v. Now let $D := V - \{u, w, z\}$. By the choice of u, w and z, $\langle V - D \rangle$ has no isolated vertex. Also, since every vertex in D is adjacent to v, $\langle D \rangle$ has no isolated vertex. Moreover, the vertices u, w and z all are dominated by v. Therefore, D is a TRDS of G. In this case, $\gamma_{tr}(G) \leq n - 3$, a contradiction. ■

The next corollary follows readily from Theorems 4 and 5.

Corollary 4. *If G is a connected graph of order $n \geq 8$ and minimum degree $\delta \geq 2$, then $\gamma_{tr}(G) \leq n - 3$.*

Remark 3. *Since $\gamma_{tr}(C_7) = 5$, the condition of Corollary 4 cannot be violated. Also, the example in Figure 2 shows that this upper bound is sharp.*

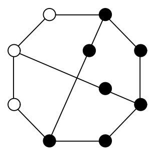

Figure 2: Graph G of order $n = 10$, where $\gamma_{tr}(G) = n - 3$.

Let G be a graph of order n and minimum degree $\delta \geq 2$. A vertex of degree greater than two is called a *large vertex.* We denote the set of vertices of degree two in G by $S(G)$ and the set of large vertices in G by $L(G)$. If there is no confusion, we denote these two sets by S and L, respectively. Also, let $\mathcal{F}$ be the family of connected graphs G such that $L(G)$ is an independent set.

Theorem 6. *If G is a connected graph of order $n \geq 11$ and minimum degree $\delta \geq 2$, then $\gamma_{tr}(G) \leq n-4$.*

Proof. Let $e = uv \in E$, where $u, v \in L$, and let $G' = G - e$. Observe that $\delta(G') \geq 2$ and $\gamma_{tr}(G') \geq \gamma_{tr}(G)$.

Assume that G' is disconnected. Then G' has exactly two components, say G_1 and G_2. Suppose that $u \in V(G_1)$ and $v \in V(G_2)$. Two cases arise.

(1) $\delta(G_1) = n_1 - 1$ or $\delta(G_2) = n_2 - 1$, where $n_i = n(G_i)$, $i = 1, 2$. We may assume that $\delta(G_1) = n_1 - 1$. Then G_1 is a complete graph. Now if $\delta(G_2) = n_2 - 1$, then $\{u, v\}$ is a TRDS of G; else, by Theorem 3, $\gamma_{tr}^M(G_2) \leq n_2 - 2$. Hence, $\mathrm{dom_{tr}}(v, G_2) \leq n_2 - 2$ and there is a TRDS of G_2, say D, containing v of order at most $n_2 - 2$. It can be seen that $D \cup \{u\}$ is a TRDS of G of order at most $n - 4$. So, $\gamma_{tr}(G) \leq n - 4$.

(2) $\delta(G_1) < n_1 - 1$ and $\delta(G_2) < n_2 - 1$. Hence, by Theorem A, we have $\gamma_{tr}(G_1) \leq n_1 - \delta(G_1)$ and $\gamma_{tr}(G_2) \leq n_2 - \delta(G_2)$. Thus, $\gamma_{tr}(G) \leq \gamma_{tr}(G') = \gamma_{tr}(G_1) + \gamma_{tr}(G_2) \leq n - \delta(G_1) - \delta(G_2) \leq n - 4$.

Assume that G' is connected. Since $\delta(G') \geq 2$ and $\gamma_{tr}(G) \leq \gamma_{tr}(G')$, we can apply the same argument for edges joining two large vertices in G'. Hence, we may assume that there is no edge in G joining two large vertices; that is L is an independent set.

Also, if G has a cut vertex, then by Theorem 4, $\gamma_{tr}(G) \leq n - 4$. Thus, we may further assume that G is a 2-connected graph.

Claim **1**. *If the induced subgraph $\langle S \rangle$ has more than one component of size at least two, then $\gamma_{tr}(G) \leq n - 4$.*

Proof of Claim **1**. Assume that uv and wz are two edges in two distinct components of $\langle S \rangle$, and let $D := V - \{u, v, w, z\}$. Since $\delta \geq 2$ and the degree of every vertex in $V - D$ is one, every vertex in $V - D$ has a neighbor in D. If a vertex x is adjacent to the vertices u and v, then x is a cut vertex, which contradicts the 2-connectivity of G. Similarly, the vertices w and z have no common neighbor. Also by assumption the edges uv and wz are in distinct components. Therefore, no vertex in $D \cap S$ is adjacent to two vertices of $V - D$. Thus, the subgraph $\langle D \cap S \rangle$ has no isolated vertex. Moreover, since every vertex of L is of degree at least three, the subgraph $\langle D \cap L \rangle$ has no isolated vertex. Hence, D is a TRDS of G of order $n - 4$ and we are done. $\square$

From now on, we may assume that $\langle S \rangle$ has at most one nontrivial component. Also, if G is a cycle, then by Theorem B(ii), we are done. Thus, assume further that $\Delta \geq 3$. Note that every component of S is a path, as $\Delta(\langle S \rangle) \leq 2$ and G is a connected graph.

Claim **2**. *If the nontrivial component of $\langle S \rangle$ has more than 4 vertices, then $\gamma_{tr}(G) \leq n-4$.*

Proof of Claim **2**. Assume that a component of $\langle S \rangle$ is a path of order at least five, say $uxy \dots wz$. Let v be the neighbor of u in L. It can be seen that the set $D := V - \{u, v, w, z\}$ is a TRDS of G of size $n-4$ and we are done. □

Claim **3**. *If there exist a pair of vertices in L, with no common neighbor in S, then $\gamma_{tr}(G) \leq n-4$.*

Proof of Claim **3**. Let u and v be two vertices in L with no common neighbor in S. By Claim 1, every vertex in S, which is not in the unique nontrivial component of S is an isolated vertex in $\langle S \rangle$. Since the unique nontrivial component of S is a path, $\langle S \rangle$ has at most two vertices of degree one. Thus, each of the vertices u and v has at least one neighbor in S, which is isolated vertex in $\langle S \rangle$. We call these isolated vertices w and z, respectively. Then, it is easy to see that $D := V - \{u, v, w, z\}$ is a TRDS of size $n-4$, and hence $\gamma_{tr}(G) \leq n-4$. □

If $\Delta = n-1$, then let v be a vertex of degree $n-1$ and u be a vertex with minimum degree. If $\{u, v\}$ is not a TRDS of G, then there is a vertex w in G in which $N_G(w) = \{u, v\}$. Thus $deg_G(w) = 2$, and by choosing u, we have $deg_G(u) = 2$ and v is a cut vertex which contradicts the 2-connectivity of G. Hence, $\gamma_{tr}(G) = 2 \leq n-4$.

If $\Delta \geq 5$, then by Theorem B, $\gamma_{tr}(G) \leq n-4$. Hence, suppose that $\Delta \leq 4$.

First, assume $\Delta = 4$ and v be a vertex of degree 4 with neighbors u, w, z, y. Since L is an independent set, the neighbors of v are of degree 2. Moreover, the set $\{u, w, z, y\} \subseteq S$ is independent, for otherwise, v is a cut vertex. Let $D := V - \{u, v, w, z\}$. The only neighbor of v in D is y, which is not adjacent to u, w and z; so y is not isolated in $\langle D \rangle$. Thus, if x is an isolated vertex in $\langle D \rangle$, then $N_G(x) \subseteq \{u, w, z\}$. But in this case v is a cut vertex, a contradiction. Hence, D is a TRDS of order $n-4$.

Assume that $\Delta = 3$. Let p be the number of edges in E having both of their ends in S. By Claims 1 and 2, $p < 4$. Now by double counting the number of edges with one end in S and the other end in L, we have $3|L| = 2|S| - 2p$, where $|S| + |L| = n$. Thus $5|L| = 2n - 2p$, and so $5|(2n - 2p)$.

If $n = 11$, then $p = 1$. Hence, $|L| = 4$ and $|S| = 7$. Let uv be the unique edge in S. Suppose that w and z are the neighbors of u and v in L, respectively. If $w = z$, then w is a cut vertex, a contradiction. Thus, $w \neq z$. Let y be a vertex in $L - \{w, z\}$ and x be a neighbor of y in S. Since $p = 1$, the set $D := V - \{u, v, x, y\}$ is a TRDS of G of order $n - 4$. Thus, $\gamma_{tr}(G) \leq n - 4$.

If $n = 12$, then $p = 2$. Hence, $|L| = 4$ and $|S| = 8$. By Claim 1, we have a component in S which is a path, say uvw. Suppose that every two vertices in L have a common neighbor in S (note that this neighbor is unique with respect to pairs, because every vertex in S is of degree two). Each of the vertices u, v and w has at most one neighbor in L and hence, is not a common neighbor for the vertices in L. Thus, $|S| \geq \binom{|L|}{2} + 3$, i.e., $8 \geq \binom{4}{2} + 3$, a contradiction. Hence, there exist a pair of vertices in L, with no common neighbor in S. By Claim 3, we have $\gamma_{tr}(G) \leq n - 4$.

Now we prove the theorem by induction on n. For $n = 11$ and 12, we have shown that $\gamma_{tr}(G) \leq n - 4$.

Assume that $n = n(G) > 12$ and the inequality holds for every graph H with $n(H) < n$. Let u and v be two adjacent vertices in S. Let w and z be the other neighbors of u and v, respectively. If $w = z$, then w is a cut vertex, a contradiction. Hence, $w \neq z$. Let H be the connected graph obtained from G by deleting the vertices u and v and joining the vertices w and z. We have $\delta(H) \geq 2$ and $11 \leq n(H) = n - 2 < n$. By the induction hypothesis, $\gamma_{tr}(H) \leq n(H) - 4 = n - 6$. Suppose that D' is a TRDS of H with at most $n - 6$ vertices. If $w \in D'$ or $z \in D'$, then $D := D' \cup \{u, v\}$ is a TRDS of G of order at most $n - 4$. Assume that $w, z \notin D'$. If $D' \cup \{u, v\}$ is not a TRDS of G, then at least one of the vertices w or z, say w, has no neighbor in $V(H) - D'$ other than z, i.e., every neighbor of w in G other than u is in D'. Thus, $D := D' \cup \{u, w\}$ is a TRDS of G of order at most $n - 4$.

Hence, we may assume that S is an independent set; i.e, $p = 0$. If every two vertices in L have a common neighbor in S (which is unique with respect to pair), then $|S| \geq \binom{|L|}{2}$, where $|S| = \frac{3}{5}n$ and $|L| = \frac{2}{5}n$, which implies that $n \leq 10$, a contradiction. We thus conclude that G satisfies the condition of Claim 3, and hence, $\gamma_{tr}(G) \leq n - 4$. ■

Remark 4. *The graph of Figure 2 shows that the conditions of Theorem 6 cannot be violated. Also, since $\gamma_{tr}(C_{11}) = 7$, the upper bound is sharp.*

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