On b-coloring of cartesian product of graphs

RAMIN JAVADI and BEHNAZ OMOOMI

Department of Mathematical Sciences
Isfahan University of Technology
84156-83111, Isfahan, Iran

Abstract

A b-coloring of a graph G by k colors is a proper k-coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ color classes. The b-chromatic number $\varphi(G)$ of a graph G is the maximum k for which G has a b-coloring by k colors. This concept was introduced by R.W. Irving and D.F. Manlove in 1999. In this paper we study the b-chromatic numbers of the cartesian products of paths and cycles with complete graphs and the cartesian product of two complete graphs.

Key Words: b-chromatic number, b-coloring, dominating coloring.

1 Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A *proper k-coloring* of graph G is a function c defined on the $V(G)$, onto a set of colors $C = \{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. In fact, for every i, $1 \leq i \leq k$, the set $c^{-1}(\{i\})$ is an independent set of vertices which is called a *color class*. The minimum cardinality k for which G has a proper k-coloring is the *chromatic number* $\chi(G)$ of G.

A *b-coloring* of a graph G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other $k-1$ color classes. We will call such a vertex x_i, a *b-dominating vertex* and the set of vertices $\{x_1, x_2, \ldots, x_k\}$ a *b-dominating system*. The *b-chromatic number* $\varphi(G)$ of a graph G is the maximum k for which G has a b-coloring by

k colors. The b-chromatic number was introduced by R.W. Irving and D.F. Manlove in [2]. They proved that determining $\varphi(G)$ is NP-hard for general cases, but it is polynomial for trees. An immediate and useful bounds for $\varphi(G)$ is:

$$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1, \tag{1}$$

where $\Delta(G)$ is the maximum degree of vertices in G.

The cartesian product of two graphs G_1 and G_2, denoted by $G_1 \Box G_2$, is a simple graph with $V(G_1) \times V(G_2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \Box G_2$ if and only if either $u_1 = u_2$ and v_1, v_2 are adjacent in G_2, or u_1, u_2 are adjacent in G_1 and $v_1 = v_2$. In the sequel, where $|V(G_1)| = m$ and $|V(G_2)| = n$, we consider the vertex set of the graph $G_1 \Box G_2$, as an $m \times n$ array in which the entry (i, j) corresponds to the vertex (i, j), $i \in V(G_1)$ and $j \in V(G_2)$, and each column induces a copy of graph G_1 and each row induces a copy of graph G_2. In Section 3, where $G_2 = C_n$, the neighbors of entry (i, j) in the row i are entries $(i, j \pm 1)$. In Section 4, where $G_2 = P_n$, the neighbors of entry (i, j) in the row i are entries $(i, j \pm 1)$, for $2 \leq j \leq n - 2$ and for $j = 1$ and $j = n$ are $(i, 2)$ and $(i, j - 1)$, respectively. So through this paper all first components of entries are modulo $|V(G_1)| = m$ and all second components of entries are modulo $|V(G_2)| = n$.

The b-chromatic number of the cartesian product of some graphs such as $K_{1,n} \Box K_{1,n}$, $K_{1,n} \Box P_k$, $P_n \Box P_k$, $C_n \Box C_k$ and $C_n \Box P_k$ was studied in [3]. In this paper we study the b-chromatic numbers of the cartesian products of paths and cycles with complete graphs and the cartesian product of two complete graphs.

2 b-chromatic number of graph $K_m \Box G$

In this section we present some results on the b-chromatic number of the cartesian product of the complete graphs with every graph G.

Proposition 1. *Let c be a b-coloring of graph $K_m \Box G$ by φ colors, where $\varphi > m$, and $v \in V(G)$. Then the column corresponding to the vertex v, contains at most $deg_G(v)$ b-dominating vertices.*

Proof. By assumption $\varphi > m$, therefore in the b-coloring c there is at least one color that does not appear in the column corresponding to the vertex v of G, we denote this column by K_m^v. On the other hand this missing color must appear in the neighbors of all b-dominating vertices in K_m^v, which are obviously in different columns. Therefore the number of b-dominating vertices in K_m^v is at most $deg_G(v)$. $\square$

If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph G with n vertices, then by Proposition 1, in graph $K_m \square G$ each column, denoted by $K_m^{(i)}$, $1 \leq i \leq n$, contains at most d_i b-dominating vertices. Therefore, every b-dominating system of G contains at most $\sum_{i=1}^{n} d_i$ vertices. So we have the following upper bounds for $\varphi(K_m \square G)$ which improves the given upper bounds in [3].

Corollary 1. *If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of graph G with n vertices and e edges, then*

$$\varphi(K_m \square G) \leq \sum_{i=1}^{n} d_i = 2e.$$

Now we prove a lemma on completing a partial proper coloring of graph $K_m \square G$ for every graph G. A *partial proper coloring* of a graph is an assignment of colors to some vertices of G, such that the adjacent vertices receive different colors.

Let $S_1, \ldots, S_n$ be some sets. A *system of distinct representatives* (SDR) for these sets is an n-tuple $(x_1, \ldots, x_n)$ of elements with the properties that $x_i \in S_i$ for $i = 1, \ldots, n$ and $x_i \neq x_j$ for $i \neq j$. It is a well known theorem that the family of sets S_i has an SDR if and only if it satisfies the Hall's condition, which is for every subset $I \subseteq \{1, 2, \ldots, n\}$, $| \cup_{i \in I} S_i| \geq |I|$, [1].

Lemma 1. *Let G be a graph and m be a positive integer, which $m \geq 2\Delta(G)$. If c is a partial proper coloring of graph $K_m \square G$ by m colors, such that each column has no uncolored vertices or at least $2\Delta(G)$ uncolored vertices, then c can be extended to a proper coloring of graph $K_m \square G$ by m colors.*

Proof. In a partial proper coloring of graph $K_m \square G$ by m colors, consider a column with $k \geq 1$ uncolored vertices $v_1, v_2, \ldots, v_k$, where by assumption $k \geq$

$2\Delta(G)$. Without loss of generality we denote k missing colors by $1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$, let S_i be the set of colors that can be used to color the vertex v_i, properly, so $S_i \subseteq \{1, 2, \ldots, k\}$. For extending this coloring to a proper coloring of this column, it is enough to find an SDR for the family of sets S_i, $1 \leq i \leq k$. For this purpose we show that the family of sets S_i, $1 \leq i \leq k$, satisfies the Hall's condition. Let $I \subseteq \{1, 2, \ldots, k\}$, which $|I| = r$.

If $r \leq \Delta(G)$, then for some $i_0 \in I$ we have

$$| \cup_{i \in I} S_i| \geq |S_{i_0}| \geq k - \Delta(G) \geq \Delta(G) \geq r = |I|.$$

If $r > \Delta(G)$, then $\cup_{i \in I} S_i = \{1, 2, \ldots, k\}$. Because if a color say i_0, $1 \leq i_0 \leq k$, does not appear in any set S_i, $i \in I$, then each vertex v_i, $i \in I$, has a neighbor say u_i of color i_0 in the row containing v_i. Since all of the vertices u_i have the same color, they are in different columns. Hence we must have $r = |I| \leq \Delta(G)$, which is a contradiction. Therefore

$$| \cup_{i \in I} S_i| = k \geq |I|.$$

So the coloring of each column can be extended and the proof is completed. $\square$

Proposition 2. *For every two graphs G and H, if graph H' is obtained by replacing one of the edges of H with a path of length 3, then $\varphi(G \square H') \geq \varphi(G \square H)$.*

Proof. Let $e = xy$ be an edge in H and H' be obtained by replacing e with the path $xwzy$. Moreover, assume that c is a b-coloring of graph $G \square H$ by $\varphi(G \square H)$ colors. We define a b-coloring c' of graph $G \square H'$ as follows. We color the vertices in the columns corresponding to the vertices w and z in H' the same as the color of vertices in the columns y and x in the coloring c, respectively. Finally we color the rest of the vertices the same as the coloring c. It is easy to see that c' is a proper coloring and the b-dominating system in c is a b-dominating system in c'. $\square$

Corollary 2. *For every positive integers m, n,*

$$\varphi(K_m \square C_{n+2}) \geq \varphi(K_m \square C_n) \quad \text{and} \quad \varphi(K_m \square P_{n+2}) \geq \varphi(K_m \square C_n).$$

Proof. Let $\varphi(K_m \square C_n) = k$. The graph C_{n+2} is obtained by replacing one edge $e = xy$ in C_n by the path $xwzy$. So by Proposition 2, there is a b-coloring c of graph $K_m \square C_{n+2}$ by k colors. Furthermore by the proof of Proposition 2, we see that there is no b-dominating vertex in the columns corresponding to the vertices w and z in the coloring c. Thus c is also a b-coloring of graph $K_m \square P_{n+2}$, where P_{n+2} is obtained by deleting the edge wz in C_{n+2}. $\square$

3 b-chromatic number of graph $K_m \square C_n$

In this section we determine the exact value of $\varphi(K_m \square C_n)$. We know that $\chi(K_m \square C_n) = m$ and $\Delta(K_m \square C_n) = m+1$. Therefore by (1),

$$m \leq \varphi(K_m \square C_n) \leq m+2. \tag{2}$$

To prove our main theorem in this section, we need the following lemma.

Lemma 2. *If c is a b-coloring of graph $K_m \square C_n$ by k colors and S is a b-dominating system in c, such that:*
(i) *there is one b-dominating vertex, say (r,s), $r \neq m$, in a color class x, such that the vertices (r,s) and $(r, s \pm 1)$ are not in S,*
(ii) *row m have no vertex in S,*
(iii) *when n is odd, $c(m, s-1) \neq x$.*
Then $\varphi(K_{m+1} \square C_n) \geq k+1$.

Proof. Without loss of generality we assume that $(r,s) = (1,1)$. We present a b-coloring c' of graph $K_{m+1} \square C_n$ by $k+1$ colors as follows:

$$c'(i,j) = \begin{cases} x & \text{if } \ (i,j) = (m+1, 1), \\ k+1 & \text{if } \ (i,j) = (1,1), \\ k+1 & \text{if } \ (i,j) = (m+1, 2t), \ 1 \leq t \leq \lfloor \frac{n}{2} \rfloor, \\ c(m, 2t-1) & \text{if } \ (i,j) = (m+1, 2t-1), \ 2 \leq t \leq \lceil \frac{n}{2} \rceil, \\ k+1 & \text{if } \ (i,j) = (m, 2t-1), \ 2 \leq t \leq \lceil \frac{n}{2} \rceil, \\ c(i,j) & \text{otherwise.} \end{cases}$$

From the definition of c' and the property (iii) it is easy to see that c' is a proper coloring. Moreover, because of the properties (i), (ii) and since in coloring c' each column has a vertex with color $k+1$, every vertex in S is a b-dominating vertex in c'. Also the vertex $(1,1)$ is a b-dominating vertex with color $k+1$. Therefore c' is a b-dominating coloring by $k+1$ colors. $\square$

Theorem 1. *For positive integers $m, n \geq 4$:*

$$\varphi(K_m \square C_n) = \begin{cases} m & \text{if } m \geq 2n, \\ m+1 & \text{if } m = 2n-1, \\ m+2 & \text{if } m \leq 2n-2. \end{cases}$$

Proof. Assume $m \geq 2n$. By Corollary 1, $\varphi(K_m \square C_n) \leq 2n$. Hence by (2), we have $\varphi(K_m \square C_n) = m$.

Now let $m = 2n-1$, by Corollary 1, $\varphi(K_m \square C_n) \leq 2n = m+1$. To prove the equality we present a b-coloring of graph $K_m \square C_n$ by $m+1$ colors.

Consider an $(m+1) \times n$ array and fill some of the entries of this array as follows. We denote this partial proper coloring by c. All second components of entries are modulo n, $1 \leq j \leq n$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $r = 0, 1$.

$$\begin{aligned} &c(2\lceil \tfrac{j}{2} \rceil - r, j) = 2j - r, \\ &c(2k, 2k-2) = 4k-1, \; c(2k, 2k+1) = 4k-3, \\ &c(m+1, 2k-r) = 4k+2r-3. \end{aligned}$$

If n is odd, then we also define

$$c(m+1, n) = c(n, n-1) = c(n+1, 1) = 4.$$

In Figure 1, this array with the filled entries for $n = 4$ is shown.

It is not hard to see that, this array with some filled entries is a partial proper coloring of graph $K_{m+1} \square C_n$, which each column has three filled entries. Since $m = 2n-1 \geq 7$, every column has at least 4 uncolored vertices. Hence by Lemma 1, c can be extended to a proper coloring of graph $K_{m+1} \square C_n$ by $m+1$ colors. Now to obtain the desired coloring, we delete the last row. Note that in this coloring of graph $K_m \square C_n$, each column has exactly one missing color. The set of vertices $\{\ (2\lceil j/2 \rceil - r, j) \mid 1 \leq j \leq n, r = 0, 1\}$ is a b-dominating system.

1	3		
2	4	1	3
		5	7
5	7	6	8
3	1	7	5

Figure 1: A partial proper coloring of graph $K_8 \Box C_4$.

Because for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, the missing color of column $2k$ is $4k-3$ which is the color of vertices $(2k, 2k+1)$ and $(2k-1, 2k-1)$ and the missing color of column $2k-1$ is $4k-1$ which is the color of vertices $(2k, 2k-2)$ and $(2k-1, 2k)$.

Now assume $9 \leq m \leq 2n-2$; by (2), $\varphi(K_m \Box C_n) \leq m+2$. To show the equality, we present a b-coloring of graph $K_m \Box C_n$ by $m+2$ colors. Consider an $(m+2) \times n$ array and fill some of the entries of this array as follows. We denote this partial proper coloring by c. All second components of entries are modulo n and the values are modulo $m+2$, $1 \leq j \leq \lceil m/2 \rceil + 1$, $1 \leq k \leq \lceil \frac{m}{4} \rceil$ and $r = 0, 1$.

$$c(2\lceil \tfrac{j}{2} \rceil - r, j) = 2j - r,$$
$$c(2k-r, 2k-2) = 4k+r-1,\ \ c(2k-r, 2k+1) = 4k+r-3,$$
$$c(m+1, 2k-r) = 4k+2r-3,\ \ c(m+2, 2k-r) = 4k+2r-2.$$

If $m \equiv 0, 3 \pmod 4$, then we also define

$$c(\lceil m/2 \rceil + 2 - r, \lceil m/2 \rceil) = 6 - r,$$
$$c(\lceil m/2 \rceil + 2 - r, \lceil m/2 \rceil + 2) = 5 + r,$$
$$c(m+1+r, \lceil m/2 \rceil + 1) = 6 - r.$$

In Figure 2, this array with the filled entries for $m = 9$ and $n = 6$ is shown.

It is not hard to see that, this array with some filled entries is a partial proper coloring of graph $K_{m+2} \Box C_n$, which each column has four filled entries. Since $m \geq 9$, every column has at least 4 uncolored vertices. Hence by Lemma 1, c can be extended to a proper coloring of graph $K_{m+2} \Box C_n$ by $m+2$ colors. Now to

1	3	2			4
2	4	1			3
	8	5	7	6	
	7	6	8	5	
10			1	9	11
9			11	10	1
3	1	7	5	11	9
4	2	8	6	1	10

Figure 2: A partial proper coloring of graph $K_{11}\Box C_6$.

obtain the desired coloring, we delete the last two rows. Note that in this coloring of graph $K_m\Box C_n$, each column has exactly two missing colors. Similarly, it is not hard to see that the set of vertices $\{\ (2\lceil j/2\rceil - r, j) \mid 1 \le j \le \lceil m/2\rceil + 1, r = 0, 1\}$ is a b-dominating system. Because for $1 \le k \le \lceil \frac{m}{4}\rceil$, the missing colors of column $2k$ are $4k-3$ and $4k-2$, while we have $c(2k, 2k+1) = c(2k-1, 2k-1) = 4k-3$ and $c(2k-1, 2k+1) = c(2k, 2k-1) = 4k-2$. Moreover, the missing colors of column $2k-1$ are $4k-1$ and $4k$, while we have $c(2k, 2k-2) = c(2k-1, 2k) = 4k-1$ and $c(2k-1, 2k-2) = c(2k, 2k) = 4k$.

Now assume $4 \le m \le 8$ and $m \le 2n-2$. In Figure 3 we provide a b-coloring of graphs $K_4\Box C_n$, $n = 4, 5$ and $K_7\Box C_n$, $n = 5, 6$. In these colorings the b-dominating system, S is the set of circled vertices. Then we apply Lemma 2 for the given coloring of $K_4\Box C_4$ twice, first for $(r, s) = (3, 4)$ and second for $(r, s) = (2, 3)$. Also, we apply that lemma for the given coloring of graph $K_4\Box C_5$, twice, first for $(r, s) = (3, 4)$ and second for $(r, s) = (3, 4)$. Thus we obtain the desired b-colorings of graphs $K_m\Box C_n$, $m = 5, 6$, $n = 4, 5$. Moreover, we apply Lemma 2 for the given colorings of graphs $K_7\Box C_5$ and $K_7\Box C_6$ for $(r, s) = (6, 5)$ and obtain the desired b-colorings of graphs $K_8\Box C_n$, $n = 5, 6$. By Corollary 2, to obtain a b-coloring of graph $K_m\Box C_n$, $n \ge t$, it is enough to have a b-coloring of graphs $K_m\Box C_t$ and $K_m\Box C_{t+1}$. Therefore, from the b-coloring obtained above we have the desired b-coloring of graphs $K_m\Box C_n$, $4 \le m \le 9$ and $m \le 2n-2$. $\Box$

1	3	5	6
2	6	**2**	3
5	4	1	**4**
4	2	4	2

1	3	5	2	6
2	6	4	6	3
5	4	1	**5**	4
4	2	**6**	3	5

3	2	5	9	1
4	1	4	8	2
7	5	9	4	8
6	3	8	6	7
8	4	1	2	5
9	6	7	3	**4**
5	8	2	7	6

1	3	2	3	7	4
2	4	1	4	8	3
7	8	5	7	6	9
9	7	6	8	5	8
5	9	4	1	9	2
6	5	9	2	**3**	1
8	6	3	9	4	7

Figure 3: A b-coloring of graphs $K_4 \square C_n$, $n = 4, 5$ and $K_7 \square C_n$, $n = 5, 6$.

4 b-chromatic number of graph $K_m \square P_n$

In this section, by using the results of Section 2, we determine the exact value of $\varphi(K_m \square P_n)$. We know that $\chi(K_m \square P_n) = m$ and $\Delta(K_m \square P_n) = m+1$. Therefore by (1),

$$m \leq \varphi(K_m \square P_n) \leq m + 2. \tag{3}$$

Theorem 2. *For positive integers $m, n \geq 4$:*

$$\varphi(K_m \square P_n) = \begin{cases} m & \text{if } m \geq 2n-2, \\ m+1 & \text{if } 2n-5 \leq m \leq 2n-3, \\ m+2 & \text{if } m \leq 2n-6. \end{cases}$$

Proof. Assume $m \geq 2n - 2$. By Corollary 1, $\varphi(K_m \square P_n) \leq 2(n-1)$. Hence by (3), $\varphi(K_m \square P_n) = m$.

If $\varphi(K_m \square P_n) = m + 2$, then there is not any b-dominating vertex in the first and the last columns of graph $K_m \square P_n$, because the vertices in the first and the last columns are of degree m. Furthermore, by Proposition 1, the other $n - 2$

columns each contains at most two b-dominating vertices. Therefore, $m+2 = \varphi(K_m \Box P_n) \le 2(n-2)$. Hence for $m \ge 2n-5$, we have $\varphi(K_m \Box P_n) \le m+1$.

Now let $2n-5 \le m \le 2n-3$, we present a b-coloring of graph $K_m \Box P_n$ by $m+1$ colors. We consider two cases.

Case 1. $m = 2n-3$.

We define a coloring $c : V(K_m \Box P_n) \to \{1, 2, \ldots, m+1\}$ by:

$$c(i,j) = \begin{cases} m-1 & \text{if} \quad (i,j) = (m,1), \\ m+1 & \text{if} \quad (i,j) = (3j-4, j), \ 1 \le j \le n-1, \\ m+1 & \text{if} \quad (i,j) = (3n-6, n), \\ i+j-1 \pmod{m} & \text{otherwise.} \end{cases}$$

It is not hard to see that the above assignment is a proper coloring of graph $K_m \Box P_n$. In fact this assignment presents a partial circular latin rectangle with the rest entries filled as above.

The set $S = \{(m-1, 1), (3n-5, n), (3j-5, j), (3j-3, j) \mid 2 \le j \le n-1\}$ (the summations are modulo m) is a b-dominating system. Obviously, each vertex dominates $m-1$ neighbors on its column, which are in different color classes. So for a vertex to be a b-dominating vertex it is enough to dominate a vertex with the color which is missed in its column. The missing color in column j, $2 \le j \le n-1$ is $4j-5$, in column 1 is m and in column n is $4n-7$. Moreover, we have $c(m-1, 2) = m$, $c(3n-5, n-1) = 4n-7$, $c(3j-5, j+1) = 4j-5$, and $c(3j-3, j-1) = 4j-5$. Therefore, the set S is b-dominating system of colors $\{1, 2, \ldots, m+1\}$. In Figure 5(a), this coloring is shown for $m = 5$, where the circled vertices are b-dominating vertices.

Now let $m = 2n-5$, consider a b-coloring of graph $K_m \Box P_{n-1}$ by $m+1$ colors as above. We add a column and color it the same as column 1. This yields a b-coloring of graph $K_m \Box P_n$ by $m+1$ colors.

Case 2. $m = 2n-4$.

As illustrated in Figure 4, $\varphi(K_4 \Box P_4) = 5$, the b-dominating vertices are circled.

5	2	5	4
1	3	4	1
3	5	1	2
4	1	2	3

Figure 4: A b-coloring of graph $K_4 \square P_4$ by 5 colors.

Assume $n \geq 5$, we define the coloring $c : V(K_m \square P_n) \rightarrow \{1, 2, \ldots, m+1\}$ by:

$$c(i,j) = \begin{cases} m-1 & \text{if } \ (i,j) = (m,1), \\ m+1 & \text{if } \ (i,j) = (3j-4, j), \ 1 \leq j \leq \lceil \frac{n}{2} \rceil, \\ m+1 & \text{if } \ (i,j) = (3j-5, j), \ \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-1, \\ m+1 & \text{if } \ (i,j) = (3n-7, n), \\ i+j-1 \pmod m & \text{otherwise.} \end{cases}$$

It is not hard to see that, the assignment above is a proper coloring of graph $K_m \square P_n$. Similar to Case 1, it can be easily checked that the set $\{(m-1,1), (3n-6, n), (3j-5, j), (3j-3, j), (i, 3i-6), (i, 3i-4) \mid \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-1, \ 2 \leq j \leq \lceil \frac{n}{2} \rceil\}$ (the summations in the first components are modulo m and in the second components are modulo n) is a b-dominating system. In Figure 5(b) this coloring is shown for $m = 6$, which the circled vertices are b-dominating vertices.

1	2	3	6
2	6	4	5
3	4	5	1
6	5	1	2
4	1	6	3

(a)

1	2	3	7	5
2	7	4	5	7
3	4	5	6	1
4	5	6	1	2
7	6	7	2	3
5	1	2	3	4

(b)

Figure 5: A b-coloring of graphs $K_5 \square P_4$ and $K_6 \square P_5$ by 6 and 7 colors.

Now assume $m \leq 2n-6$, and let $n' = n-2$. Since $m \leq 2n'-2$, by Theorem 1, $\varphi(K_m \square C_{n'}) = m+2$, $n' \geq 4$. Hence by Corollary 2, $\varphi(K_m \square P_n) \geq m+2$. Therefore by (3), $\varphi(K_m \square P_n) = m+2$, for $n \geq 6$.

For $n = 5$ a b-coloring of graph $K_m \square P_n$ is shown in Figure 6, the b-dominating vertices are circled. $\square$

5	1	6	4	6
6	2	5	3	1
4	3	4	2	4
1	4	1	6	5

Figure 6: A b-coloring of graph $K_4 \square P_5$ by 6 colors.

5 b-chromatic number of graph $K_n \square K_n$

We know that $\chi(K_n \square K_n) = n$ and $\Delta(K_n \square K_n) = 2n-2$. So by (1), $n \leq \varphi(K_n \square K_n) \leq 2n-1$. In this section we improve these bounds and prove that $2n-3 \leq \varphi(K_n \square K_n) \leq 2n-2$. Finally we provide a conjecture that $\varphi(K_n \square K_n) = 2n-3$, $n \geq 5$.

Lemma 3. *Let c be a b-coloring of graph $K_n \square K_n$ by $2n-1$ colors. If two vertices (i,j) and (i,t) are b-dominating vertices in the b-coloring c, then in columns j and t there are no other b-dominating vertices.*

Proof. Let c be a b-coloring of graph $K_n \square K_n$ by $2n-1$ colors. It is obvious that if a vertex (x,y) is a b-dominating vertex in the b-coloring c, then all its $2n-2$ neighbors must have different colors. So the colors of the vertices in the row x and the column y are different. Now, assume to the contrary that the vertices (i,j), (i,t) and (i',j), $i' \neq i$, are b-dominating vertices. Since the vertex (i,t) is a b-dominating vertex, the vertices in row i and column t all have different colors. Therefore, if $c(i',t) = a$, then no vertex in row i has color a. On the other hand the vertex (i,j) is a b-dominating vertex, hence in column j we must have a vertex with color a. Now, in both row i' and column j we have vertices by color a. It contradicts our assumption that the vertex (i',j) is a b-dominating vertex. By the same reason the vertex (i',t), for $i' \neq i$, is not b-dominating vertex. $\square$

Theorem 3. *For every positive integer $n \geq 2$, we have*

$$\varphi(K_n \square K_n) \leq 2n-2.$$

Proof. We know that $\varphi(K_n \square K_n) \leq 2n-1$. Let $\varphi(K_n \square K_n) = 2n-1$ and c be a b-coloring by $2n-1$ colors. Without loss of generality we assume that rows 1 to r each has at least two b-dominating vertices and rows $r+1$ to n each has at most one b-dominating vertex. Moreover, without loss of generality, we assume that the b-dominating vertices in the first r rows are in the first s columns. By Lemma 3, in each column j, $1 \leq j \leq s$, there is only one b-dominating vertex. If $r = 0$ or $s = n$, then we have at most n b-dominating vertices which is a contradiction. The size of the b-dominating system in coloring c is at most $s + (n - r)$. Now if $r > 0$ and $s < n$, then the number of b-dominating vertices is at most $s + (n - r) \leq 2n - 1 - r < 2n - 1$ which also contradicts our assumption. $\square$

Theorem 4. *For every positive integer $n \geq 5$, we have*

$$\varphi(K_n \square K_n) \geq 2n - 3.$$

Proof. We present a b-coloring c by $2n-3$ colors, for two cases n odd and n even. First, we define a function $f : \mathbb{N} \to \mathbb{Z}$ by:

$$f(x) = \begin{cases} x & x \text{ is odd,} \\ x - 2 & x \text{ is even.} \end{cases}$$

Case 1. n is odd.

In this case we define the assignment $c : V(K_n \square K_n) \to \mathbb{N}$ by:

$$c((i,j)) = \begin{cases} i + j - 1 \pmod{n-1} & i \leq j \leq n - i - 1, \\ f(i+j) \pmod{n-1} & n - i \leq j \leq n - 2, i \leq j, \\ (i + j - 2 \pmod{n-2}) + (n-1) & j < i \leq n - 1, \\ & (i,j) \neq (n-1, n-2) \\ n - 3 & (i,j) = (n-1, n-2). \end{cases}$$

For columns $n-1$, n and row n, the assignment c is as follows.

$$c((i, n-1)) = \begin{cases} 2i - 2 \pmod{n-1} & 1 \leq i \leq \frac{n-1}{2}, \\ 2i - 1 \pmod{n-1} & \frac{n+1}{2} \leq i \leq n - 2, \\ 2n - 4 & i = n - 1. \end{cases}$$

$$c((i, n)) = \begin{cases} (2i - 2 \pmod{n-2}) + (n-1) & i \text{ odd, } i \leq n - 2, \\ i - 2 \pmod{n-1} & i \text{ even, } i \leq n - 2, \\ n - 2 & i = n - 1. \end{cases}$$

$$c((n,j)) = \begin{cases} j-1 \pmod{n-1} & j \text{ odd},\ j \leq n-3, \\ (2j-2 \pmod{n-2}) + (n-1) & j \text{ even}, \\ 2n-5 & j = n-2, \\ 1 & j = n. \end{cases}$$

The assignment c is a b-coloring and the set $S = \{(i,i),(j+1,j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n-2\}$ is a b-dominating system. Because the vertices in S all have different colors and for each vertex in S the colors in its row and columns all have different colors except two entries. As an example such a coloring for $n = 7$ is illustrated in Figure 7, the b-dominating vertices are circled.

1	2	3	4	5	6	11
7	3	4	5	1	2	6
8	9	5	1	6	4	10
9	10	11	6	3	1	2
10	11	7	8	2	3	9
11	7	8	9	4	10	5
6	8	2	7	9	11	1

Figure 7: A b-coloring of graphs $K_7 \square K_7$ by 11 colors.

Case 2. n is even.

In this case we define the assignment $c : V(K_n \square K_n) \to \mathbb{N}$ by:

$$c((i,j)) = \begin{cases} i+j-2 \pmod{n-2} & i+1 \leq j \leq n-i-1, \\ f(i+j-1) \pmod{n-2} & n-i \leq j \leq n-2, i+1 \leq j, \\ (i+j-1 \pmod{n-1}) + (n-2) & j \leq i \leq n-1, \\ & (i,j) \neq (n-1,n-2) \\ n-4 & (i,j) = (n-1,n-2). \end{cases}$$

For columns $n-1$, n and row n, the assignment c is as follows.

$$c((i,n-1)) = \begin{cases} 2i-2 \pmod{n-2} & 1 \leq i \leq \frac{n-2}{2}, \\ 2i-1 \pmod{n-2} & \frac{n}{2} \leq i \leq n-3, \\ 2n-5 & i = n-2, \\ 2n-4 & i = n-1, \\ n-3 & i = n. \end{cases}$$

$$c((i,n)) = \begin{cases} (2i \pmod{n-1}) + (n-2) & i \text{ odd},\ i \leq n-2, \\ i-2 \pmod{n-2} & i \text{ even},\ i \leq n-2, \\ n-3 & i = n-1, \\ 1 & i = n. \end{cases}$$

$$c((n,j)) = \begin{cases} (2j-2 \pmod{n-1}) + (n-2) & j \text{ odd},\ 3 \leq j \leq n-3, \\ j-2 \pmod{n-2} & j \text{ even},\ j \leq n-3 \\ n-4 & j = 1, \\ 2n-5 & j = n-2. \end{cases}$$

The assignment c is a b-coloring and the set $S = \{(i,i),(j-1,j) \mid 1 \leq i \leq n-1, 2 \leq j \leq n-2\} \cup \{(n-1,n-2)\}$ is b-dominating system. Because the vertices in S all have different colors and for each vertex in S the colors in its row and columns all have different colors except two entries. As an example such a coloring for $n = 8$ is illustrated in Figure 8, the b-dominating vertices are circled. $\square$

7	1	2	3	4	5	6	8
8	9	3	4	5	1	2	6
9	10	11	5	1	6	4	12
10	11	12	13	6	3	1	2
11	12	13	7	8	2	3	9
12	13	7	8	9	10	11	4
13	7	8	9	10	4	12	5
4	6	10	2	7	11	5	1

Figure 8: A b-coloring of graphs $K_8 \square K_8$ by 13 colors.

Remark. For $n = 3$ the only way to have a b-coloring by 4 colors is Figure 9(a), with the circled vertices as b-dominating vertices; which is impossible, so $\varphi(K_3 \square K_3) = 3$. For $n = 4$ there is a b-coloring of graph $K_4 \square K_4$ by $2n-2 = 6$ colors, see Figure 9(b).

Finally, we propose the following conjecture.

Conjecture 1. *For every positive integer $n \geq 5$, $\varphi(K_n \square K_n) = 2n-3$.*

1	2	3
4	3	1
2	4	?

(a)

1	2	3	4
5	4	1	2
6	5	4	3
3	6	2	1

(b)

Figure 9: A partial b-coloring of graphs $K_3 \square K_3$ and $K_4 \square K_4$.

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