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On b-coloring of the Kneser graphs ☆

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ABSTRACT

A b-coloring of a graph G by k colors is a proper k-coloring of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ color classes. The b-chromatic number of a graph G, denoted by $\varphi(G)$, is the maximum k for which G has a b-coloring by k colors. It is obvious that $\chi(G) \leq \varphi(G)$. A graph G is b-continuous if for every k between $\chi(G)$ and $\varphi(G)$ there is a b-coloring of G by k colors. In this paper, we study the b-coloring of Kneser graphs $K(n, k)$ and determine $\varphi(K(n, k))$ for some values of n and k. Moreover, we prove that $K(n, 2)$ is b-continuous for $n \geq 17$.

1. Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A *proper k-coloring* of G is a function c defined from $V(G)$ onto a set of colors $C = \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \leq i \leq k$, the set $c^{-1}(i)$ is a nonempty independent set of vertices which is called *color class* i. The minimum cardinality k for which G has a proper k-coloring is the *chromatic number* of G, denoted by $\chi(G)$.

A *b-coloring* of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other $k-1$ color classes. Such a vertex x_i is called a *b-dominating vertex*, and the set of vertices $\{x_1, x_2, \ldots, x_k\}$ is called a *b-dominating system*. The *b-chromatic number* of G, denoted by $\varphi(G)$, is the maximum k for which G has a b-coloring by k colors. It is an elementary exercise to observe that every proper coloring with $\chi(G)$ colors is a b-coloring. The b-chromatic number was introduced by R.W. Irving and D.F. Manlove in [4]. (See also [5,6].)

Immediate and useful bound for $\varphi(G)$ is:

$$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1, \tag{1}$$

where $\Delta(G)$ is the maximum degree of vertices in G.

The graph G is *b-continuous* if for every k between $\chi(G)$ and $\varphi(G)$ there is a b-coloring with k colors. A peculiar characteristic of b-coloring is that not all graphs are b-continuous. For example, the 3-dimensional cube Q_3 is not b-continuous: $\chi(Q_3) = 2$ and $\varphi(Q_3) = 4$, but Q_3 has no b-coloring with three colors [4]. Only a few classes of graphs are known to be b-continuous [1,3].

Let $S = \{1, 2, \ldots, n\}$ and let V be the set of all k-subsets of S, where $k \leq \frac{n}{2}$. The *Kneser graph* with parameters n and k, denoted by $K(n, k)$, is the graph with vertex set V such that two vertices are adjacent if and only if the corresponding subsets are disjoint. It is known that $\chi(K(n, k)) = n - 2k + 2$ [8]. In this paper, we study b-coloring of Kneser graphs. We determine $\varphi(K(2k + 1, k))$ for every k and $\varphi(K(n, 2))$ for every n. Also, we prove that $K(n, 2)$ is b-continuous for $n \geq 17$.

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2. Steiner triple systems

In this section, we recall some necessary definitions and constructions of Steiner triple systems which will be used in the proofs of our main theorems.

A *quasigroup* of order n is a pair $(Q, \circ)$, where Q is a set of size n and "$\circ$" is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. A quasigroup $(Q, \circ)$ with $Q = \{1, 2, \ldots, n\}$ is said to be *idempotent* if $i \circ i = i$, for $1 \le i \le n$ and *commutative* if $i \circ j = j \circ i$, for all $1 \le i, j \le n$. A quasigroup $(Q, \circ)$ with $Q = \{1, 2, \ldots, 2n\}$ is said to be *half-idempotent* if for $1 \le i \le n$, $i \circ i = (n \circ i) \circ (n \circ i) = i$. A quasigroup $(Q', \circ)$, where $Q' \subseteq Q$, is called a sub-quasigroup of quasigroup $(Q, \circ)$.

Example 1. Let $n = 2k + 1$ and consider the additive group $(\mathbb{Z}_n, +)$. Since n is odd, for each $i, j \in \mathbb{Z}_n$ where $i \neq j$, we have $2i \neq 2j$. Therefore, there is a permutation σ on the set $\{1, 2, \ldots, n\}$ such that for each $i \in \mathbb{Z}_n$, $\sigma(2i) = i$. Now we define the quasigroup $(Q_1, \circ)$ where $Q_1 = \mathbb{Z}_n$ and $i \circ j = \sigma(i + j)$ for every $i, j \in Q_1$. This quasigroup is an idempotent commutative quasigroup.

Let $n = 2k$ and consider the additive group $(\mathbb{Z}_n, +)$. In this case for each i, $1 \le i \le k$, $i + i = (i + k) + (i + k) = 2i$. We consider a permutation σ on the set $\{1, 2, \ldots, n\}$ such that for each i, $1 \le i \le k$, $\sigma(2i) = i$. Now we define the quasigroup $(Q_2, \circ)$ where $Q_2 = \mathbb{Z}_n$ and $i \circ j = \sigma(i+j)$ for every $i, j \in Q_2$. This quasigroup is a half-idempotent commutative quasigroup.

A *design* with parameters $t - (n, k, \lambda)$ is an ordered pair $(S, \mathcal{B})$, where S is a set of n *points* or *symbols* and $\mathcal{B}$ is a family of k-subsets of S called *blocks*, such that every t elements of S occur together in exactly λ blocks of $\mathcal{B}$. When $\lambda = 1$, it is called a *Steiner system*, and when $k = 3$, it is called a *triple system*. A design with parameters $t = 2$, $k = 3$ and $\lambda = 1$ with n points is called a *Steiner triple system of order n*, denoted by $STS(n)$.

It is known that a Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6$ [7].

2.1. *The Bose Construction:* $n \equiv 3 \pmod 6$

Let $n = 6k + 3$ and $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k + 1$ and define $S = Q \times \{1, 2, 3\}$. We denote an ordinary element of S by x_i, where $x \in Q$ and $i \in \{1, 2, 3\}$ and define $\mathcal{B}$ to contain the following two types of triples:

Type 1 : for $1 \le i \le 2k + 1$, $\{i_1, i_2, i_3\} \in \mathcal{B}$,
Type 2 : for $1 \le i < j \le 2k + 1$, $\{i_1, j_1, (i \circ j)_2\}, \{i_2, j_2, (i \circ j)_3\}, \{i_3, j_3, (i \circ j)_1\} \in \mathcal{B}$.

Then $(S, \mathcal{B})$ is a Steiner triple system of order $6k + 3$ [7].

2.2. *The Skolem Construction:* $n \equiv 1 \pmod 6$

Let $n = 6k+1$ and $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2k$ and define $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We denote an ordinary point in $Q \times \{1, 2, 3\}$ by x_i, where $x \in Q$ and $i \in \{1, 2, 3\}$ and define $\mathcal{B}$ as follows:

Type 1 : for $1 \le i \le k$, $\{i_1, i_2, i_3\} \in \mathcal{B}$,
Type 2 : for $1 \le i \le k$, $\{\infty, (k + i)_1, i_2\}, \{\infty, (k + i)_2, i_3\}, \{\infty, (k + i)_3, i_1\} \in \mathcal{B}$,
Type 3 : for $1 \le i < j \le 2k$, $\{i_1, j_1, (i \circ j)_2\}, \{i_2, j_2, (i \circ j)_3\}, \{i_3, j_3, (i \circ j)_1\} \in \mathcal{B}$.

Then $(S, \mathcal{B})$ is a Steiner triple system of order $6k + 1$ [7].

Above we have constructed Steiner triple systems of all orders $n \equiv 1, 3 \pmod 6$. Although no $STS(6k + 5)$ exists, we can get very close.

A *pairwise balanced design* or simply *PBD* is an ordered pair $(S, \mathcal{B})$, where S is a finite set of points and $\mathcal{B}$ is a collection of subsets of S called blocks, such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. When $|S| = n$ it is denoted by $PBD(n)$.

For all $n \equiv 5 \pmod 6$, we produce a *PBD* of order n with one block of size 5 and others of size 3, called 3-blocks.

2.3. *The* $n = 6k + 5$ *Construction*

Let $(Q, \circ)$ be an idempotent commutative quasiqroup of order $2k + 1$ and α be the permutation $(1, 2)(3, 4) \ldots (2k - 1, 2k)(2k + 1)$. Let $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$, we denote an ordinary point in $Q \times \{1, 2, 3\}$ by x_i, where $x \in Q$ and $i \in \{1, 2, 3\}$. Now define $\mathcal{B}$ to contain the following blocks:

Type 1 : $\{\infty_1, \infty_2, (2k + 1)_1, (2k + 1)_2, (2k + 1)_3\} \in \mathcal{B}$,
Type 2 : for $1 \le i \le k$, $\{\infty_1, (2i - 1)_1, (2i - 1)_2\}, \{\infty_1, (2i - 1)_3, (2i)_1\}, \{\infty_1, (2i)_2, (2i)_3\}$,
$\{\infty_2, (2i - 1)_2, (2i - 1)_3\}, \{\infty_2, (2i)_1, (2i)_2\}, \{\infty_2, (2i - 1)_1, (2i)_3\} \in \mathcal{B}$,
Type 3 : for $1 \le i < j \le 2k + 1$, $\{i_1, j_1, (i \circ j)_2\}, \{i_2, j_2, (i \circ j)_3\}, \{i_3, j_3, (\alpha(i \circ j))_1\} \in \mathcal{B}$.

Then $(S, \mathcal{B})$ is a $PBD(6k + 5)$ with exactly one block of size 5 and all others of size 3 [7].

For results in later sections we need some steiner triple systems containing another Steiner triple system, called subsystem.

Theorem A (*[2]*).

(i) *For every two integers* $n, m \equiv 1, 3 \pmod 6$ *such that* $n \geq 2m + 1$, *there is an* $STS(n)$ *containing a subsystem* $STS(m)$.
(ii) *For every two integers* $n, m \equiv 5 \pmod 6$ *such that* $n \geq 2m + 1$, *there is a* $PBD(n)$ *which contains a* $PBD(m)$.

A *Steiner quasigroup* $(Q, \circ)$ is a commutative quasigroup, where $i \circ i = i$ and $(i \circ j) \circ j = i$, for every $i, j \in Q$ [2].

Given a Steiner triple system, we can construct a steiner quasigroup by setting $x \circ y = z$ when $\{x, y, z\}$ is a block of the design or when $x = y = z$. Also given a *PBD* with one block of size 5 and others of size 3 and an idempotent commutative quasigroup of order 5, $(Q', \circ')$, we can construct an idempotent commutative quasigroup by setting $x \circ y = z$ when $\{x, y, z\}$ is a 3-block of the *PBD* or when $x = y = z$; and $x \circ y = x \circ' y$ when x, y are both in the block of size 5. Thus we have the following proposition.

Proposition 1. *For every odd integer* n, $n \neq 5$, *there exists an idempotent commutative quasigroup of order* n *containing a sub-quasigroup of order* 3.

3. *b*-chromatic number of the Kneser graph

In this section, we determine $\varphi(K(2k + 1, k))$ for every k and $\varphi(K(n, 2))$ for every n.

Theorem 1. *For every integer* $k \geq 3$,

$$\varphi(K(2k + 1, k)) = k + 2.$$

Proof. We know that $\Delta(K(2k + 1, k)) = k + 1$, so by Inequality (1), $\varphi(K(2k + 1, k)) \leq k + 2$. To prove the equality we describe a b-coloring of $K(2k + 1, k)$ by $k + 2$ colors as follows. For i, $1 \leq i \leq k$, we define the color class i to contain the set of vertices

$$\{\{k + 1, k + 2, \ldots, 2k + 1\} \setminus \{k + i\}\} \cup \{\{1, 2, \ldots, k\} \setminus \{i\} \cup \{k + j\} \mid 1 \leq j \leq k + 1, j \neq i\},$$

the color class $k + 1$ contains the set of vertices

$$\{k + 1, k + 2, \ldots, 2k\} \cup \{\{1, 2, \ldots, k\} \setminus \{j\} \cup \{k + j\} \mid 1 \leq j \leq k\}$$

and the color class $k + 2$ contains the set $\{\{1, 2, \ldots, k\}\}$.

Now we complete the coloring as follows. Let $A \subseteq \{1, 2, \ldots, 2k + 1\}$ be a vertex distinct from the vertices in the color classes above. If $2k + 1 \in A$ then we choose an integer $i \in A^c \cap \{1, 2, \ldots, k\}$ and add A to the color class i. If $2k + 1 \notin A$ and $2k \in A$ then we choose an integer $i \in A^c \cap \{1, 2, \ldots, k\}$, $i \neq k$, and add A to the color class i. If $2k, 2k + 1 \notin A$ then we add A to the color class $k + 2$. It is not hard to see that the vertices in each class have mutually nonempty intersections. Hence, such a coloring is a proper coloring.

In this proper coloring the set of vertices $\{\{k + 1, k + 2, \ldots, 2k + 1\} \setminus \{k + i\} \mid 1 \leq i \leq k + 1, \{1, 2, \ldots, k\}\}$ is a b-dominating system. Because, the vertex $\{1, 2, \ldots, k\}$ is adjacent to all vertices $\{k + 1, k + 2, \ldots, 2k + 1\} \setminus \{k + i\}$, $1 \leq i \leq k + 1$. Moreover, for a fixed integer i_0, $1 \leq i_0 \leq k + 1$, the vertex $\{k + 1, k + 2, \ldots, 2k + 1\} \setminus \{k + i_0\}$ is adjacent to the vertices $\{1, 2, \ldots, k\}$ and $\{1, 2, \ldots, k\} \setminus \{i\} \cup \{k + i_0\}$, $1 \leq i \leq k$, $i \neq i_0$ and for $1 \leq i_0 \leq k$, this vertex is adjacent to the vertex $\{1, 2, \ldots, k\} \setminus \{i_0\} \cup \{k + i_0\}$. □

In the sequel, we are going to determine $\varphi(K(n, 2))$. First we mention some facts, terminology and lemmas which will be used in the proof of the main theorem.

Fact 1. By the definition of $STS(n)$, it is obvious that every Steiner triple system of order n is in fact an edge decomposition of the complete graph K_n into triangles.

Fact 2. Each vertex in $K(n, 2)$ which is a 2-subset of the set $\{1, 2, \ldots, n\}$ corresponds to an edge in the complete graph K_n with vertex set $\{1, 2, \ldots, n\}$. Hence, two vertices of $K(n, 2)$ are nonadjacent if and only if the corresponding edges in K_n are adjacent.

Fact 3. If A is an independent set of vertices in $K(n, 2)$, then either all vertices in A have a common element, say a, or $A = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, for some $a, b, c \in \{1, 2, \ldots, n\}$. In other words an independent set of vertices in $K(n, 2)$ corresponds to a star subgraph with center a or a triangle subgraph in K_n. From now on we call the independent set (color class) in $K(n, 2)$ of the first form *starlike* with center a and the second form *triangular*. Moreover, for simplicity we denote the independent set $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ with $\{a, b, c\}$. Since every proper coloring is a partition of vertices into independent sets of vertices, we can consider every proper coloring of $K(n, 2)$ as an edge decomposition of the complete graph K_n into star and triangle subgraphs.

A set of vertices S is called a *dominating set*, whenever every vertex not in S has a neighbor in S. A dominating set S in G is called an *independent dominating set* when the vertices in S are mutually nonadjacent. The following proposition is a fact about dominating sets in Kneser graphs.

Proposition 2. *Let* $S = \{1, 2, \ldots, n\}$. *If* T *is a subset of* S *of size* $2k - 1$, *then the set of all* k*-subsets of* T *is an independent dominating set in the* $K(n, k)$.

Proof. Let $T \subseteq S = \{1, 2, \ldots, n\}$, $|T| = 2k - 1$ and A be a vertex in $K(n, k)$ for which $A \nsubseteq T$. So $|A \cap T| \leq k - 1$ and there is a k-subset of T, say B, for which $A \cap B = \emptyset$. Therefore, the vertices A and B are adjacent in $K(n, k)$. Obviously, every two k-subsets of T intersect, so they are not adjacent in $K(n, k)$. The statement follows. □

By the proposition above, when a Steiner system with some special parameters exists, we can find a lower bound for the b-chromatic number of $K(n, k)$.

Theorem 2. *If $(S, \mathcal{B})$ is a $k - (n, 2k - 1, 1)$ Steiner system, then $\varphi(K(n, k)) \geq |\mathcal{B}|$.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{|\mathcal{B}|}\}$. For each i, $1 \leq i \leq |\mathcal{B}|$, we define the set of all k-subsets of B_i as the color class i. Since $|B_i| = 2k - 1$, by Proposition 2, each class i is an independent set of vertices, so this partition is a proper coloring of $K(n, k)$. Moreover, by Proposition 2, each class i is a dominating set. Therefore, each element in a color class j has neighbors in all the other color classes. Hence, this partition is a b-coloring of $K(n, k)$ by $|\mathcal{B}|$ colors. □

Lemma 1. *Assume that c is a proper coloring of $K(n, 2)$ and $A_1, A_2, \ldots, A_t$, $|A_i| \geq 3$, $1 \leq i \leq t$, are the starlike color classes in c, with centers $a_1, a_2, \ldots, a_t$, respectively. Then c is a b-coloring of $K(n, 2)$ if and only if the following conditions hold.*

(i) *$a_1, a_2, \ldots, a_t$ are distinct,*
(ii) *every 2-subset of the set $\{a_1, a_2, \ldots, a_t\}$ is in $\cup_{k=1}^{t} A_k$, and*
(iii) *for each i, $1 \leq i \leq t$, there exists an element $x_i \notin \{a_1, a_2, \ldots, a_t\}$, where $\{a_i, x_i\} \in A_i$.*

Proof. Assume that c is a b-coloring of $K(n, 2)$. Suppose that $a_i = a_j$ for some $i \neq j$. Hence, $A_i \cup A_j$ is an independent set in $K(n, 2)$. This means that no vertex in the color class A_i has a neighbor in the color class A_j, which contradicts that c is a b-coloring. So $a_i \neq a_j$ for all $1 \leq i \neq j \leq t$.

Now consider an arbitrary 2-subset $\{a_i, a_j\}$ of the set $\{a_1, a_2, \ldots, a_t\}$. If $\{a_i, a_j\} \notin \cup_{k=1}^{t} A_k$, then this vertex is in a triangular color class, say $\{a_i, a_j, b\}$. In this color class, the vertices $\{a_i, a_j\}$ and $\{a_i, b\}$ are not b-dominating vertices because they have no neighbor in the color class A_i. The vertex $\{a_j, b\}$ also is not a b-dominating vertex since it has no neighbor in the color class A_j. This is a contradiction. Thus $\{a_i, a_j\} \in \cup_{k=1}^{t} A_k$, for all i, j. Since in each starlike color class A_i we must have a b-dominating vertex, the property (iii) is obviously concluded.

Now assume that c is a proper coloring of $K(n, 2)$ that satisfies (i), (ii) and (iii). It is enough to show that in each color class of c, there is a b-dominating vertex. In the starlike color classes A_i, $1 \leq i \leq t$, the vertex $\{a_i, x_i\}$ is a b-dominating vertex, because in each color class A_j, $j \neq i$, there exists a vertex $\{a_j, y\}$ such that $y \neq a_i, x_i$. Moreover, by Proposition 2 each triangular color class is a dominating set. Therefore, the vertex $\{a_i, x_i\}$ has neighbors in all color classes. On the other hand for each triangular color class $\{a, b, c\}$, by (ii), we have $|\{a, b, c\} \cap \{a_1, a_2, \ldots, a_t\}| \leq 1$. Hence there exists at least two elements, say a and b, with $a, b \notin \{a_1, a_2, \ldots, a_t\}$. Since $|A_i| \geq 3$, the vertex $\{a, b\}$ has neighbors in all starlike color classes. Furthermore, by Proposition 2 each triangular color class is a dominating set. So the vertex $\{a, b\}$ is a b-dominating vertex. □

Proposition 3. *If $n \equiv 5 \pmod 6$ then $\varphi(K(n, 2)) \geq \frac{n(n-1)}{6} - \frac{1}{3}$.*

Proof. If $n \equiv 5 \pmod 6$ then by the $6k + 5$ construction given in Section 2, we have a $PBD(n)$ with one block of size 5, say $\{1, 2, 3, 4, 5\}$, and 3-blocks otherwise. In this construction, number of 3-blocks is $\frac{n(n-1)}{6} - \frac{10}{3}$. Now we provide a b-coloring of $K(n, 2)$. We consider each 3-block as a triangular color class and define the other color classes as $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$, $\{\{2, 3\}, \{2, 4\}, \{2, 5\}\}$, and $\{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$. This is an edge decomposition of the complete graph K_n into stars and triangles, so by Fact 3 this is a proper coloring of $K(n, 2)$. Furthermore, this coloring satisfies the conditions of Lemma 1 and so is a b-coloring of $K(n, 2)$. Hence

$$\varphi(K(n, 2)) \geq \frac{n(n-1)}{6} - \frac{10}{3} + 3 = \frac{n(n-1)}{6} - \frac{1}{3}. \quad \square$$

Theorem 3. *For every positive integer n, $n \neq 8$, we have:*

$$\varphi(K(n, 2)) = \begin{cases} \left\lfloor \dfrac{n(n-1)}{6} \right\rfloor & \text{if } n \text{ is odd}, \\ \left\lfloor \dfrac{(n-1)(n-2)}{6} \right\rfloor + 3 & \text{if } n \text{ is even}. \end{cases}$$

Proof. We prove the theorem for two cases n is even and n is odd.

Case 1. n is even.

First we find an upper bound for $\varphi(K(n, 2))$. Let c be a b-coloring of $K(n, 2)$ by φ colors and t starlike color classes with centers $1, \ldots, t$ of sizes $n_1, \ldots, n_t$, respectively. Then,

$$|V(K(n, 2))| = \binom{n}{2} = \sum_{i=1}^{t} n_i + 3(\varphi - t). \quad (2)$$

By Fact 3, the coloring c corresponds to an edge decomposition of the complete graph K_n into stars and triangles. For every vertex $i \in V(K_n)$, the number of edges incident to i in the triangles of the decomposition is even. Since n is even, there is an edge incident to i in a star subgraph in the decomposition. Therefore, for each i satisfying $t+1 \le i \le n$ there is a vertex in $K(n,2)$ containing i in the starlike color classes 1 to t. Moreover, by Lemma 1, every 2-subset of the set $\{1,2,\ldots,t\}$ is in the starlike color classes. Therefore, we have

$$\sum_{i=1}^{t} n_i \ge (n-t) + \frac{t(t-1)}{2} = n + \frac{t(t-3)}{2}.$$

Hence,

$$\binom{n}{2} \ge n + \frac{t(t-9)}{2} + 3\varphi.$$

So

$$\varphi \le \frac{n(n-3)}{6} - \frac{t(t-9)}{6}.$$

The minimum of $t(t-9)$ occurs in $t=4$ and $t=5$. Therefore,

$$\varphi \le \left\lfloor \frac{n(n-3)}{6} + \frac{10}{3} \right\rfloor = \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 3. \tag{3}$$

Now we find a lower bound for $\varphi(K(n,2))$.

Case 1.1. $n = 6k$.

We consider an $STS(6k-3)$ with the Bose construction. As shown in Section 2, in this construction there are $2k-1$ disjoint blocks of Type 1. We denote these blocks by $\{a_1,b_1,c_1\},\{a_2,b_2,c_2\},\ldots,\{a_{2k-1},b_{2k-1},c_{2k-1}\}$. By Fact 1, this STS is an edge decomposition of the complete graph K_{n-3} into triangles. Now we add three new points a,b,c and then construct a proper coloring of $K(n,2)$ by $\varphi_0 = \frac{n(n-3)}{6} + 3$ colors or equivalently an edge decomposition of the complete graph K_n into φ_0 stars and triangles.

We consider every block of Type 2 in the $STS(6k-3)$ as one triangular color class. The other color classes are defined as follows. Color class A consists of

$$\{a,c_1\},\{a,c_2\},\ldots,\{a,c_{2k-1}\},\{a,b\}.$$

Color class B consists of

$$\{b,a_1\},\{b,a_2\},\ldots,\{b,a_{2k-1}\},\{b,c\}.$$

Color class C consists of

$$\{c,b_1\},\{c,b_2\},\ldots,\{c,b_{2k-1}\},\{c,a\}.$$

Also for each i, $1 \le i \le 2k-1$, we define three triangular color classes

$$\{a,a_i,b_i\},\{b,b_i,c_i\},\{c,c_i,a_i\}.$$

In the $STS(6k-3)$ the number of blocks is $\frac{(n-3)(n-4)}{6}$, of which $2k-1 = \frac{n-3}{3}$ blocks are of Type 1. Therefore, the number of color classes in the given coloring above are $\frac{(n-3)(n-4)}{6} - \frac{n-3}{3} + 3 + 3\frac{(n-3)}{3} = \frac{n(n-3)}{6} + 3 = \varphi_0$.

For $n=6$, it is obvious that this coloring is a b-coloring of $K(6,2)$ by 6 colors. For $k \ge 2$, we have only three starlike color classes and this coloring satisfies the conditions of Lemma 1. Hence, the given coloring is a b-coloring of $K(n,2)$. Therefore, $\varphi \ge \frac{n(n-3)}{6} + 3 = \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 3$.

Case 1.2. $n = 6k+2$, $k \ge 2$, or $n = 6k+4$.

We consider an $STS(n-1)$ with the Bose or the Skolem construction given in Section 2. Moreover, in this construction we consider three disjoint blocks $\{a,b,c\}$, $\{a',b',c'\}$, and $\{a'',b'',c''\}$ in which $\{a,a',a''\}$ is a block. Now we add a new point d and construct a b-coloring of $K(n,2)$ by $\varphi_0 = \frac{(n-1)(n-2)}{6} + 3$ colors as follows.

We consider every block in $STS(n-1)$ except four blocks $\{a,b,c\}$, $\{a',b',c'\}$, $\{a'',b'',c''\}$, and $\{a,a',a''\}$ as a color class. Moreover, we add the following color classes. Color class A consists of $\{a,b\},\{a,c\},\{a,a'\}$. Color class B consists of $\{a',b'\},\{a',c'\},\{a',a''\}$. Color class C consists of $\{a'',b''\},\{a'',c''\},\{a'',a\}$. Color class D consists of $\{d,x\}$, $x \notin \{b,b',b'',c,c',c''\}$. Finally, we add three triangular color classes $\{b,c,d\}$, $\{b',c',d\}$ and $\{b'',c'',d\}$. The number of these color classes is $\varphi_0 = \frac{(n-1)(n-2)}{6} - 4 + 4 + 3 = \frac{(n-1)(n-2)}{6} + 3$.

We have only four starlike color classes and this coloring satisfies the conditions of Lemma 1. Hence, the given coloring is a b-coloring of $K(n,2)$. Therefore, $\varphi \ge \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 3$.

Case 2. n is odd.

First we find an upper bound for $\varphi(K(n,2))$. Let c be a b-coloring of $K(n,2)$ by $\varphi = \varphi(K(n,2))$ colors and t starlike color classes with centers $1,\ldots,t$ of sizes $n_1,\ldots,n_t$, respectively. Then,

$$|V(K(n,2))| = \binom{n}{2} = \sum_{i=1}^{t} n_i + 3(\varphi - t). \tag{4}$$

By Lemma 1, every 2-subset of the set $\{1,2,\ldots,t\}$ is in the color classes 1 to t. Moreover, in the color class i we must have a b-dominating vertex, say $\{i,x\}$, where $x \in \{t+1,t+2,\ldots,n\}$. Hence,

$$\sum_{i=1}^{t} n_i \geq \frac{t(t-1)}{2} + t = \frac{t(t+1)}{2}.$$

Therefore,

$$\binom{n}{2} \geq 3\varphi + \frac{t(t+1)}{2} - 3t = 3\varphi + \frac{t(t-5)}{2}.$$

So

$$\varphi \leq \frac{n(n-1)}{6} - \frac{t(t-5)}{6}.$$

The minimum of the expression $t(t-5)$ occurs in $t=2$ and $t=3$, so $\varphi \leq \frac{n(n-1)}{6} + 1$.

Now we prove that $\varphi \leq \frac{n(n-1)}{6}$. Suppose $\varphi = \frac{n(n-1)}{6} + 1$, hence, $t=2$ or $t=3$. For every vertex $i \in V(K_n)$, the number of edges incident to i in the triangles of the decomposition is even. Since n is odd, the number of edges incident to i in the stars of the decomposition is also even. Equivalently, in the b-coloring of $K(n,2)$ the number of vertices containing i in the starlike color classes are even numbers.

If $t=3$ then by Lemma 1 (ii) and (iii), the vertices $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ in $K(n,2)$ are in the starlike color classes with centers 1, 2, or 3 and for every i, $1 \leq i \leq 3$, there is a vertex $\{i,x\}$ in the starlike color classes which $x \neq 1,2,3$. So by the discussion above, for every i, $1 \leq i \leq 3$, at least two vertices $\{i,x\}$ and $\{i,y\}$, where $x,y \neq 1,2,3$, are in the starlike color classes. Therefore, $\sum_{i=1}^{3} n_i \geq 3 + 2\times 3 = 9$. So by Relation (4), $\binom{n}{2} \geq 9 + 3(\varphi - 3) = 3\varphi$. Hence, $\varphi \leq \frac{n(n-1)}{6}$, which contradicts our assumption.

Now let $t=2$. By Lemma 1 (ii) and (iii), the starlike color class with center 1 contains vertex $\{1,2\}$ and at least one more vertex, say $\{1,3\}$. By the discussion above, if the vertex $\{1,i\}$ in $K(n,2)$ is in the starlike color class with center 1, then the vertex $\{2,i\}$ is in the starlike color class with center 2. If the vertices $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are the only vertices in the starlike color classes, then there is no b-dominating vertex in these classes. Therefore, the starlike color class with center 1 and consequently, the starlike color class with center 2 each one contains at least more two vertices. Hence, $\sum_{i=1}^{2} n_i = 1 + 2\times 3 = 7$. Therefore, by Relation (4)

$$\binom{n}{2} \geq 7 + 3(\varphi - 2) = 3\varphi + 1.$$

So $\varphi \leq \frac{n(n-1)}{6}$, which contradicts our assumption.

Therefore, $\varphi \leq \left\lfloor \frac{n(n-1)}{6} \right\rfloor$. If $n \equiv 1, 3 \pmod 6$ then an $STS(n)$ exists. Therefore, by Theorem 2, $\varphi \geq \frac{n(n-1)}{6}$. If $n \equiv 5 \pmod 6$ then by Proposition 3, $\varphi \geq \frac{n(n-1)}{6} - \frac{1}{3}$. Hence, $\varphi = \left\lfloor \frac{n(n-1)}{6} \right\rfloor$. □

Since the Petersen graph is Kneser graph $K(5,2)$, we get the following result.

Corollary 1. *If P is the Petersen graph, then $\varphi(P) = 3$.*

Kneser graph $K(8,2)$ is an exception.

Proposition 4. $\varphi(K(8,2)) = 9$.

Proof. Consider the notations in the proof of Theorem 3 for Case 1. By Inequality (3), we have $\varphi(K(8,2)) \leq 10$ and the equality holds if and only if $t=4$ or $t=5$. Assume that a b-coloring of $K(8,2)$ exists with 10 colors and $A_1, A_2, \ldots, A_t$ are starlike color classes with centers $1,2,\ldots,t$, respectively.

If $t=4$ then by Equality (2), $\sum_{i=1}^{4} n_i = 10$. By Lemma 1 (ii) and (iii), every 2-subset of the set $\{1,2,3,4\}$ is in $\cup_{i=1}^{4} A_i$ and for each i, $1 \leq i \leq 4$, there exists $x_i \notin \{1,2,3,4\}$, where $\{i,x_i\} \in A_i$. On the other hand $n-t$ and the number of vertices containing i in triangular color classes are even numbers. So there are at least two vertices $\{i,x_i\}$, $\{i,y_i\}$ in the starlike color classes, where $x_i, y_i \notin \{1,2,3,4\}$. Hence, $\sum_{i=1}^{4} n_i = 10 \geq 6 + 4\times 2 = 14$, which is contradiction.

If $t = 5$ then by Equality (2), $\sum_{i=1}^{5} n_i = 13$. On the other hand, similar to the above by Lemma 1 (ii) and (iii), $\sum_{i=1}^{5} n_i = 13 \geq 10 + 5$, a contradiction. So $\varphi(K(8, 2)) \leq 9$.

Now we provide a b-coloring of $K(8, 2)$ by 9 colors. First we consider an $STS(7)$ and delete one point of it. What remains is a decomposition of K_6 into 4 triangles and a 1-factor called $F = \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$. Now we add two new points a and b and define the color classes as all triangles in the decomposition above in addition to the triangular color classes $\{a, a_1, b_1\}$, $\{a, a_2, b_2\}$ and $\{b, a_3, b_3\}$ and the starlike color classes $\{\{a, a_3\}, \{a, b_3\}, \{a, b\}\}$ and $\{\{b, a_1\}, \{b, b_1\}, \{b, a_2\}, \{b, b_2\}\}$. This is a proper coloring of $K(8, 2)$ satisfying the conditions of Lemma 1, so is a b-coloring by 9 colors as desired. □

By Relation (1), $\varphi(K(n, k)) \leq \Delta + 1 = \binom{n-k}{k} + 1$. Hence $\varphi(K(n, k)) = O(n^k)$. Theorems 2 and 3 motivate us to propose the following conjecture.

Conjecture 1. *For every integer k, we have $\varphi(K(n, k)) = \Theta(n^k)$.*

4. *b*-continuity of the Kneser graph $K(n, 2)$

In this section we prove that $K(n, 2)$ is b-continuous when $n \geq 17$.

Lemma 2. (a) *Let $n = 6k + 1$ or $n = 6k + 3$ and $(S, \mathcal{B})$ be an $STS(n)$. Also let T be a subset of $S = \{1, 2, \dots, n\}$ and t be the number of blocks in $\mathcal{B}$ on the points of T, such that:*

(i) $|T| = m \geq 3$,

(ii) *for each $i \in T$, there exists $j \in T$ such that the third point of the block containing both i, j is not in T.*

Then there exists a b-coloring of $K(n, 2)$ by $\varphi - (\frac{m(m-3)}{2} - 2t)$ colors, where $\varphi = \varphi(K(n, 2))$.

(b) *Let $n = 6k + 5$ and $(S, \mathcal{B})$ be a $PBD(n)$ with one block of size 5, say $\{1, 2, n, n - 1, n - 2\}$ and the others 3-blocks. Also let T be a subset of $S = \{1, 2, \dots, n\}$ and t be the number of 3-blocks in $\mathcal{B}$ on the points of T, such that:*

(i) $|T| = m \geq 3$,

(ii) $1, 2 \in T$ *and* $n - 2, n - 1, n \notin T$,

(iii) *for each $i \in T$, $i \neq 1, 2$, there exists $j \in T$ such that the third point of the 3-block containing both i, j is not in T.*

Then there exists a b-coloring of $K(n, 2)$ by $\varphi - (\frac{m(m-3)}{2} - 2t + 1)$ colors, where $\varphi = \varphi(K(n, 2))$.

Proof. Let c be the b-coloring of $K(n, 2)$ by φ colors corresponding to $STS(n)$ or $PBD(n)$ (see Theorem 2 and Proposition 3). In the case $n = 6k + 5$, we take the centers of starlike color classes as 1 and 2.

Assume $T = \{1, 2, \dots, m\}$, consider the b-coloring c and delete all triangular color classes containing a vertex $\{i, j\} \subseteq T$.

(a) Since each vertex $\{i, j\} \subseteq T$ is contained in a triangular color class and there are exactly t triangles on the points of T, the number of deleted color classes (triangles) is $\frac{m(m-1)}{2} - 3t + t$. Now we define m new color classes as follows. New color class i, $3 \leq i \leq m - 2$, contains the set of vertices $\{\{i, j\} \mid i + 1 \leq j \leq m\}$. Also new color classes $1, 2, m - 1$ and m contain respectively the sets $\{\{1, j\} \mid 2 \leq j \leq m - 2\}$, $\{\{2, j\} \mid 3 \leq j \leq m - 1\}$, $\{\{m - 1, m\}, \{m - 1, 1\}\}$ and $\{\{m, 1\}, \{m, 2\}\}$. Moreover, if a vertex $\{i, x\}$, where $i \in T$ and $x \notin T$ is in a deleted color class, then we add this vertex to the color class i. These m new color classes together with the old color classes give us a new proper coloring of $K(n, 2)$ by $\varphi - (\frac{m(m-1)}{2} - 2t) + m$ colors.

(b) Since each vertex $\{i, j\} \subseteq T$ except $\{1, 2\}$ is contained in a triangular color class and there are exactly t triangular color classes on the points of T, the number of deleted triangles is $\frac{m(m-1)}{2} - 1 - 3t + t$. Now we define $m - 2$ new color classes as follows. Color class i, $3 \leq i \leq m$, contains the set of vertices $\{\{i, j\} \mid i + 1 \leq j \leq m\} \cup \{\{i, 1\}, \{i, 2\}\}$. Moreover, if a vertex $\{i, x\}$, where $i \in T$ and $x \notin T$ is in a deleted color class, then we add this vertex to the color class i. These $m - 2$ new color classes together with the old color classes give us a new proper coloring by $\varphi - (\frac{m(m-1)}{2} - 1 - 2t) + m - 2$ colors.

The obtained colorings in (a) and (b) satisfy the conditions of Lemma 1, so they are b-colorings. □

Lemma 3. *Let $n \geq 13$ be an odd integer and let $k = \lfloor \frac{n}{6} \rfloor$. For every odd integer m, $5 \leq m \leq k + 5$ and for every integer t, $0 \leq t \leq \frac{3m-11}{2}$, where $(m, t) \neq (5, 2), (7, 5), (k + 5, 0)$, there exists an $STS(n)$ or $PBD(n)$ and a set T satisfying the conditions of Lemma 2.*

Proof. Let $l = \lfloor \frac{n}{3} \rfloor$. Depending on n, using the Bose construction, the Skolem construction or the $6k + 5$ construction given in Section 2 and the quasigroups of Example 1, construct an $STS(n)$ or a $PBD(n)$.

If $t = 0$, then it is easy to find a set T with parameters (m, t). Assume $5 \leq m \leq k + 5$ and m is odd.

(a) If $1 \leq t \leq \frac{m-5}{2}$, then define

$$T = \{l_1, i_1, (l - i)_1 \mid 1 \leq i \leq t\} \cup \{j_1 \mid t + 1 \leq j \leq m - 4 - t\} \cup \{(\sigma(l))_2, 1_3, (\sigma^{-1}(k + 2) - 1)_3\}.$$

(b) If $\frac{m-5}{2} < t < m-5$, then define

$$T = \left\{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq \frac{m-5}{2}\right\} \cup \{(\sigma(l))_2, (\sigma(2(m-5-t)))_2, (\sigma(m-5))_2, (\sigma(2l-m+5))_2\}.$$

(c) If $m-5 \leq t < 3(\frac{m-5}{2})$, then define

$$T = \left\{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq \frac{m-5}{2}\right\} \cup \{(\sigma(l))_2, (\sigma(1))_2, (\sigma(3(m-5)-2t))_2, (\sigma(2l-m+5))_2\}.$$

(d) If $3(\frac{m-5}{2}) \leq t \leq 2m-11$, then define

$$T = \left\{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq \frac{m-5}{2}\right\} \cup \{(\sigma(l))_2, (\sigma(1))_2, (\sigma(l-1))_2, (\sigma(4(m-5)-2t))_2\}.$$

The set T given above satisfies the conditions of Lemma 2 (with an appropriate renaming of elements of S). If $m \geq 11$ then $2m-11 \geq \frac{3m-11}{2}$, hence, for each $11 \leq m \leq k+5$ and $0 \leq t \leq \frac{3m-11}{2}$, we are done. Moreover, by the construction above there exists such a set T for $(m,t) = (5,0)$, $(m = 7, 0 \leq t \leq 3)$, $(m = 9, 0 \leq t \leq 7)$. For $(m,t) = (5,1)$, let $T = \{1_1, (l-1)_1, (\sigma(l))_2, 1_2, (l-1)_2\}$. For $(m,t) = (7,4)$, let $T = \{1_1, (l-1)_1, 2_1, (l-2)_1, (\sigma(l))_2, (\sigma(1))_2, (\sigma(l-1))_2\}$.

Now we construct a set T with parameters $(m,t) = (9,8)$. Since $m \leq k+5$, we have $n \geq 25$. Now if $n \equiv 1, 3 \pmod 6$, then by Theorem A there is an $STS(n)$ containing an $STS(9)$ on the set $T_0 = \{1, 2, \ldots, 9\}$. So the set $T = T_0 \cup \{10\} - \{9\}$ is the desired set with parameters $(m,t) = (9,8)$. If $n \equiv 5 \pmod 6$, then we consider an idempotent commutative quasigroup containing a sub-quasigroup of order 3 (see Proposition 1). Without loss of generality we can assume that $\{1,2,3\}$ is the sub-quasigroup of order 3. Then by applying this quasigroup to the $6k+5$ construction (see Section 2), we construct a $PBD(n)$ and define $T = \{\infty_1, \infty_2, 3_1, i_1, i_2, i_3 \mid i = 1, 2\}$. The set T is the desired set (with an appropirate renaming of elements of S). □

Lemma 4. *Let $n \geq 13$ be an odd integer and $k = \lfloor \frac{n}{6} \rfloor$. For every even integer m, $4 \leq m \leq k+5$ and every integer t, $0 \leq t \leq m-4$, there exists an $STS(n)$ or $PBD(n)$ and a set T satisfying the conditions of Lemma 2. Moreover, when $n \geq 19$ and $n \neq 6k+5$ such an STS and a set T exist for $(m,t) \in \{(6,4), (8,8)\}$,*

Proof. Let $l = \lfloor \frac{n}{3} \rfloor$. Consider the $STS(n)$ or $PBD(n)$ as in the proof of Lemma 3.

If $t = 0$, then it is easy to find a set T with parameters (m,t). Assume $4 \leq m \leq k+5$ and m is even.

(a) If $1 \leq t \leq \frac{m-4}{2}$, then define

$$T = \{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq t\} \cup \{j_1 \mid t+1 \leq j \leq m-4-t\} \cup \{(\sigma(l))_2, 1_3, (\sigma^{-1}(k+2)-1)_3\}.$$

(b) If $\frac{m-4}{2} < t < m-4$, then define

$$T = \left\{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq \frac{m-4}{2}\right\} \cup \{(\sigma(l))_2, (\sigma(2(m-4-t)))_2, (\sigma(m-4))_2\}.$$

(c) If $t = m-4$, then define

$$T = \left\{l_1, i_1, (l-i)_1 \mid 1 \leq i \leq \frac{m-4}{2}\right\} \{(\sigma(l))_2, (\sigma(1))_2, (\sigma(m-4))_2\}.$$

The set T given above satisfies the conditions of Lemma 2 (with an appropirate renaming of elements of S). Now, assume $n \geq 19$ and $n \neq 6k+5$, we construct sets T with parameters $(m,t) = (6,4), (8,8)$. By Theorem A there is an $STS(n)$ containing the $STS(7)$ on points $\{1, 2, \ldots, 7\}$. Now let $T = \{1, 2, \ldots, 6\}$, it is clear that T is a set satisfying the conditions of Lemma 2 with parameters $(m,t) = (6,4)$. Also there is an $STS(n)$ containing the $STS(9)$ on points $\{1, 2, \ldots, 9\}$. Now let $T = \{1, 2, \ldots, 8\}$, it is clear that T is a set satisfying the conditions of Lemma 2 with parameters $(m,t) = (8,8)$. □

Theorem 4. *For every integer n, $n \geq 17$, Kneser graph $K(n,2)$ is b-continuous.*

Proof. We prove the theorem for two cases n odd and n even. Let $X(n)$ be the set of numbers x for which there is a b-coloring of $K(n,2)$ by x colors.

Case 1. n is odd.

In this case we prove the theorem by induction on n. Assume for an odd integer n, $n \geq 19$, that $K(n-2,2)$ is b-continuous. Therefore, by the definition and Theorem 3, for every integer x, $n-4 \leq x \leq \left\lfloor \frac{(n-2)(n-3)}{6} \right\rfloor$, we have $x \in X(n-2)$. We consider a b-coloring of $K(n-2,2)$ with x colors and provide a b-coloring of $K(n,2)$ by $x+2$ colors. For this purpose, we add two new color classes $\{\{n,i\} \mid 1 \leq i \leq n-1\}$, $\{\{n-1,i\} \mid 1 \leq i \leq n-2\}$. This coloring satisfies the conditions of Lemma 1, so it is a b-coloring. To prove the b-continuity of $K(n,2)$ it is enough to prove $x \in X(n)$ for every integer x, $3 + \left\lfloor \frac{(n-2)(n-3)}{6} \right\rfloor \leq x \leq \left\lfloor \frac{n(n-1)}{6} \right\rfloor = \varphi$. For this purpose, let $\psi = \left\lfloor \frac{n(n-1)}{6} \right\rfloor - \left\lfloor \frac{(n-2)(n-3)}{6} \right\rfloor - 3$.

Table 1
The values are $\frac{m(m-3)}{2} - 2t + 1$.

t	m				
	3	4	5	6	7
0	1	3	6	10	–
1			4	8	13
2				6	11
3					9
4					7

Claim. *For every integer x, $1 \le x \le \psi$, we have $\varphi - x \in X(n)$.*

Proof of claim. Let $\mathcal{A}$ be the set of all positive integers x such that there exists a set $T \subseteq \{1, 2, \dots, n\}$ which satisfies the assumptions of Lemma 2 with parameters (m, t), and $\frac{m(m-3)}{2} - 2t = x$.

Case 1.1. $n = 6k + 1$ or $n = 6k + 3$, $k \ge 3$.

By Lemma 2(a), it is enough to show that for every x, $1 \le x \le \psi$, $x \in \mathcal{A}$. By Lemma 4 there exists a set T with parameters $(m, t) = (6, 4), (m, t) = (8, 8)$. Therefore, $1, 4 \in \mathcal{A}$. Moreover, by Lemma 3, for every odd integer m, $5 \le m \le k+5$, we have $\frac{m(m-3)}{2}, \frac{m(m-3)}{2} - 2, \dots, \frac{m(m-3)}{2} - (3m-11) = \frac{(m-3)(m-6)}{2} + 2 \in \mathcal{A}$. Also by Lemma 4, for every even integer m, $4 \le m \le k+5$, we have $\frac{m(m-3)}{2}, \frac{m(m-3)}{2} - 2, \dots, \frac{m(m-3)}{2} - (m-4) = \frac{(m-1)(m-4)}{2} + 2 \in \mathcal{A}$. Therefore, $1, 2, 3, 4, \dots, \frac{(k+3)k}{2} + 1 \in \mathcal{A}$. Since $\frac{(k+3)k}{2} + 1 \ge 4k - 2 \ge \psi$, we are done.

Case 1.2. $n = 6k + 5$.

By Lemma 2(b), it is enough to show that for every integer x, $0 \le x \le \psi - 1$, $x \in \mathcal{A}$. All things in Case 1.1 hold in this case as well, except the set T with parameters $(m, t) = (6, 4), (8, 8)$. So we have $\{1, 2, 3, \dots, \psi - 1\} - \{1, 4\} \subseteq \mathcal{A}$. Also there exists a set T with parameters $(m, t) = (3, 0)$ satisfying Lemma 2(b). Thus $0 \in \mathcal{A}$.

To complete the proof, we show that $\varphi - 2$ and $\varphi - 5$ are in $X(n)$. Consider the quasigroup of Example 1 and construct a $PBD(n)$ using the $6k + 5$ construction. Let c be the b-coloring of $K(n, 2)$ corresponding to this PBD by φ colors (see Proposition 3) where ∞_1, ∞_2 are the centers of the starlike color classes. Now let $T = \{\infty_1, \infty_2, (2k + 1)_1, 2_1, 1_2\}$, delete all triangular color classes containing a vertex $\{i, j\} \subseteq T$ and define 3 new starlike color classes with centers $(2k+1)_1, 2_1, 1_2$. Deleted color classes are triangles $\{(2k + 1)_1, 2_1, 1_2\}, \{\infty_1, 2_1, 1_3\}, \{\infty_2, 2_1, 2_2\}, \{\infty_1, 1_2, 1_1\}$ and $\{\infty_2, 1_2, 1_3\}$. Thus new coloring is a b-coloring by $\varphi - 5 + 3$ colors. Now let $T = \{\infty_1, \infty_2, 2_1, 2_2, 2_3, (2k+1)_2, (2k+1)_3\}$, delete all triangular color classes containing a vertex $\{i, j\} \subseteq T$ and define 5 new starlike color classes with centers $2_1, 2_2, 2_3, (2k + 1)_2, (2k + 1)_3$. Since we have deleted 10 triangular color classes, we obtain a b-coloring of $K(n, 2)$ by $\varphi - 5$ colors. So the claim is proved.

To complete the induction we need to show that $K(17, 2)$ is b-continuous. By Lemmas 3 and 4, there is a set T satisfying the conditions of Lemma 2 with parameters (m, t) shown in Table 1. The values in the table are $x = \frac{m(m-3)}{2} - 2t + 1$. Therefore, by Lemma 2(b) for the values x given in Table 1, $\varphi(K(17, 2)) - x = 45 - x \in X(17)$. Moreover, as it is proved in Cases 1.2, $\varphi(K(17, 2)) - 2$ and $\varphi(K(17, 2)) - 5$ are in $X(17)$. Hence, for every i, $34 \le i \le 45$, $i \in X(17)$.

Similarly, by Lemma 2(a) for the values x given in Table 1, $\varphi(K(15, 2)) - x - 1 = 34 - x \in X(15)$. Therefore, for every i, $25 \le i \le 35$ and $i \ne 31, 34$, $i \in X(15)$. By a similar discussion, for every i, $16 \le i \le 26$ and $i \ne 22, 25$, $i \in X(13)$. We have already proved that $x \in X(n - 2)$ implies $x + 2 \in X(n)$. Therefore, for every i, $20 \le i \le 37$ and $i \ne 26, 33$, $i \in X(17)$. By Lemma 3, for $n = 13, 15, 17$ there is a set $T \subseteq \{1, 2, \dots, n\}$ with parameters $(m, t) = (9, 8)$. Thus, by Lemma 2, $33 \in X(17)$, $24 \in X(15)$ and $15 \in X(13)$, so $26, 19 \in X(17)$. Finally, for $n = 13$ there is a set T with parameters $(m, t) = (7, 1), (9, 7)$, so $14, 13 \in X(13)$, thus $18, 17 \in X(17)$. We can easily see that $16 \in X(17)$ by constructing a b-coloring with 16 starlike color classes. This assures b-continuity of $K(17, 2)$.

Case 2. n is even.

Let $n \ge 18$ be an even integer. Then $K(n-1, 2)$ is b-continuous and $x \in X(n-1)$ holds whenever $n-3 \le x \le \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor$. Now we add a new color class $\{\{n, i\} \mid 1 \le i \le n - 1\}$ to this coloring. This is a b-coloring of $K(n, 2)$ by $x + 1$ colors. Hence $y \in X(n)$ for every integer y with $n-2 \le y \le \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 1 = \varphi - 2$. It is enough to prove $\varphi - 1 = \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 2 \in X(n)$. For this purpose, consider the b-coloring of $K(n, 2)$ by φ colors in the proof of Theorem 3. Assume that $\{a, x, y\}$ and $\{b, x, z\}$ are two triangular color classes, where a and b are the centers of some starlike color classes, A and B. We delete them and add a new starlike color class $\{\{x, y\}, \{x, z\}, \{x, a\}, \{x, b\}\}$. Finally, we add vertex $\{a, y\}$ to the starlike color class A and the vertex $\{b, z\}$ to the starlike color class B. The obtained coloring satisfies the conditions of Lemma 1 therefore, is a b-coloring of $K(n, 2)$ by $\varphi - 1$ colors. □

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