On the Locating Chromatic Number of the Cartesian Product of Graphs

Ali Behtoei[*] and Behnaz Omoomi[†]

Department of Mathematical Sciences

Isfahan University of Technology

84156-83111, Isfahan, Iran

Abstract

Let c be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple $c_\Pi(v) := (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$, where $d(v, V_i) = \min\{d(v, x) \mid x \in V_i\}, 1 \le i \le k$. If distinct vertices have distinct color codes, then c is called a locating coloring. The minimum number of colors needed in a locating coloring of G is the locating chromatic number of G, denoted by $\chi_L(G)$. In this paper, we study the locating chromatic numbers of grids, the cartesian product of paths and complete graphs, and the cartesian product of two complete graphs.

Keywords: Cartesian product, Locating coloring, Locating chromatic number.

1 Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of G, $k \in \mathbb{N}$, is a function c defined from $V(G)$

[*] *alibehtoei@math.iut.ac.ir*

[†] *bomoomi@cc.iut.ac.ir*

onto a set of colors $[k] := \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \le i \le k$, the set $c^{-1}(i)$ is a nonempty independent set of vertices which is called the color class i. The minimum cardinality k for which G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them, and for a subset S of $V(G)$, the distance between u and S is given by $d(u, S) := \min\{d(u, x) \mid x \in S\}$.

Definition. [3] *Let c be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple*

$$c_\Pi(v) := (d(v, V_1), d(v, V_2), \ldots, d(v, V_k)).$$

If distinct vertices of G have distinct color codes, then c is called a locating coloring of G. The locating chromatic number, $\chi_{_L}(G)$, is the minimum number of colors in a locating coloring of G.

The concept of locating coloring was first introduced and studied by Chartrand et al. in [3]. They established some bounds for the locating chromatic number of a connected graph. They also proved that for a connected graph G with $n \ge 3$ vertices, we have $\chi_{_L}(G) = n$ if and only if G is a complete multipartite graph. Hence, the locating chromatic number of the complete graph K_n is n. Also for paths and cycles of order $n \ge 3$ it is proved in [3] that $\chi_{_L}(P_n) = 3$, $\chi_{_L}(C_n) = 3$ when n is odd, and $\chi_{_L}(C_n) = 4$ when n is even. The locating chromatic number of trees, Kneser graphs, and the amalgamation of stars are studied in [3], [2], and [1], respectively. For more results in the subject and related subjects, see [1] to [9].

Obviously, $\chi(G) \le \chi_{_L}(G)$. Note that the i-th component of the color code of each vertex in the color class V_i is zero and its other components are non zero. Hence, a proper coloring is a locating coloring whenever the color codes of vertices in each color class are different. In a proper coloring of G, a vertex is called *colorful* if all of the colors appear in its closed neighborhood, and the color of a colorful vertex is called a *full* color. Note that in each proper m-coloring of K_m all of the vertices are colorful. We have the following observation.

Observation 1. *Let G be a connected graph. (a) In a locating coloring of G, there are no two colorful vertices that are assigned the same color. Therefore, if* *there is a locating k-coloring of G, then there are at most k colorful vertices. (b) If G contains two disjoint cliques of order k, then $\chi_{_L}(G) \geq k+1$.*

Recall that the cartesian product of two graphs G and H, denoted by $G\square H$, is a graph with vertex set $V(G)\times V(H)$ in which two vertices (a,b) and (a',b') are adjacent in it whenever $a=a'$ and $bb' \in E(H)$, or $aa' \in E(G)$ and $b=b'$. Vertices of the cartesian product $G\square H$ can be represented by an $|V(G)|$ by $|V(H)|$ array, such that the induced subgraph on the vertices of each row is isomorphic to H and the induced subgraph on the vertices of each column is isomorphic to G. In this paper, we study the locating chromatic number of the grid $P_m\square P_n$, $K_m\square P_n$ and $K_m\square K_n$.

2 The locating chromatic numbers of $P_m\square P_n$ and $K_m\square P_n$

In this section, we determine the exact value of the locating chromatic number of the grid $P_m\square P_n$ and $K_m\square P_n$. First, we give an upper bound for the locating chromatic number of the cartesian product of two arbitrary connected graphs.

Proposition. *If G and H are two connected graphs, then*

$$\chi_{_L}(G\square H) \leq \chi_{_L}(G)\ \chi_{_L}(H).$$

Proof. Let $m := \chi_{_L}(G)$ and $A_1, A_2, ..., A_m$ be the color classes of a locating m-coloring of G. Also, let $n := \chi_{_L}(H)$ and $B_1, B_2, ..., B_n$ be the color classes of a locating n-coloring of H. For each $i \in [m]$ and each $j \in [n]$, $A_i \times B_j$ is an independent set in $G\square H$. Hence, the partition $\{A_i \times B_j \mid i \in [m],\ j \in [n]\}$ of vertices of $G\square H$ can be considered as the color classes of a proper coloring of $G\square H$. To see that this is a locating coloring, let (a,b) and (a',b') be two distinct vertices in the color class $A_i \times B_j$ and, without loss of generality, assume that $a \neq a'$. Note that $d(b,B_j) = d(b',B_j) = 0$ while, by assumption, there exists $k \in [m]\setminus\{i\}$ such that $d(a,A_k) \neq d(a',A_k)$. Hence

$$\begin{aligned} d((a,b), A_k \times B_j) &= d(a,A_k) + d(b,B_j) \\ &= d(a,A_k) + 0 \\ &\neq d(a',A_k) + 0 \\ &= d((a',b'), A_k \times B_j). \end{aligned}$$

Thus, this coloring is a locating coloring. $\square$

For $G = H = K_2$, we have

$$\chi_{_L}(K_2\square K_2) = \chi_{_L}(C_4) = 4 = \chi_{_L}(K_2)\,\chi_{_L}(K_2).$$

Therefore, the above inequality is attainable. The following theorem shows that the exact value of the locating chromatic number of an m by n grid $P_m\square P_n$ is 4, while the given upper bound is 9.

Theorem 1. *If $n \geq m \geq 2$, then* $\chi_{_L}(P_m\square P_n) = 4$.

Proof. In each proper 3-coloring of $P_m\square P_n$ there exists an induced cycle C_4 with 3 colors. Hence, there are two colorful vertices on this cycle with the same color. Therefore, $\chi_{_L}(P_m\square P_n) \geq 4$.
For each $i \in [m]$ and $j \in [n]$, let $v_{i,j}$ be the vertex in the i-th row and j-th column of the grid $P_m\square P_n$, and let c be a proper 2-coloring of the bipartite graph $P_m\square P_n$ with the color set $\{1, 2\}$. Define the coloring c' as $c'(v_{1,1}) = 3$, $c'(v_{1,n}) = 4$ and $c'(v_{i,j}) = c(v_{i,j})$ else where. For each $i \in [m]$ and $j \in [n]$, we have

$$d(v_{i,j}, v_{1,1}) = i + j - 2, \quad d(v_{i,j}, v_{1,n}) = n + i - j - 1.$$

Thus, distinct vertices have distinct color codes with respect to the coloring c'. $\square$

Let $G := K_m\square P_n$. Vertices of G can be represented by an m by n array. Thus, G consists of m rows and n columns, in which the induced subgraph on the vertices of each column is isomorphic to K_m and the induced subgraph on the vertices of each row is isomorphic to P_n. Let $v_{i,j}$ be the vertex of G in the i-th row and j-th column. Hence, each coloring of G can be represented by an $m \times n$ matrix, in which its (i, j)-entry is the color of $v_{i,j}$. For the locating chromatic number of $K_m\square P_n$, the following cases are easy to check (see Theorem 1 and [3]).

(a) $\chi_{_L}(K_1\square P_1) = 1$, $\chi_{_L}(K_1\square P_2) = 2$, and $\chi_{_L}(K_1\square P_n) = 3$, $n \geq 3$.

(b) $\chi_{_L}(K_2\square P_n) = \chi_{_L}(P_2\square P_n) = 4$.

(c) $\chi_{_L}(K_m\square P_1) = \chi_{_L}(K_m) = m$.

In the following theorem, the exact value of $\chi_{_L}(K_m\square P_n)$ is computed in the remaining general case.

Lemma 1. *Let $m \geq 3$ and $n \geq 2$ be two positive integers. If there exists a locating $(m+1)$-coloring of $G := K_m \Box P_n$, then let C be its coloring matrix. Then every two consecutive columns of C have different missing colors. Moreover, if $m \geq 5$, then every two columns of C have different missing colors.*

Proof. First, let $m = 3$. Suppose on the contrary, there exist two consecutive columns C_j and C_{j+1} of C with the same missing color, say "4". Assume that $C_j = [1\ 2\ 3]^T$. The coloring is proper and hence C_{j+1} is a derangement of C_j, which implies $C_{j+1} = [3\ 1\ 2]^T$ or $C_{j+1} = [2\ 3\ 1]^T$. Without loss of generality, assume that $C_{j+1} = [3\ 1\ 2]^T$. If $j = 1$ and $C_{j'}$ is the first column which the color 4 appears in it, say in row i, then two vertices $v_{i,1}$ and $v_{i+1,2}$ ($i+1$ considered modulo m) have the same distance to the color class 4 and hence, have the same color code, which is a contradiction. For $j+1 = n$, the argument is similar. Also, when all of the columns containing the color 4 have index greeter than $j+1$, or all have index smaller than j, the argument is similar to above. Thus, assume there exist two indices j_1 and j_2, $j_1 < j < j+1 < j_2$, such that the color 4 appears in both of the columns C_{j_1} and C_{j_2}, and does not appear in the columns with indices $j_1 < k < j_2$. If $j - j_1 = j_2 - (j+1)$, then exactly four vertices in the j-th and $(j+1)$-th columns of G have the same distance $j - j_1 + 1$ to the color class 4, and hence, at least two vertices of the same color have the same color code. Thus, without loss of generality, we can assume that $j - j_1 < j_2 - (j+1)$, and the color 4 appears in the first row of the column C_{j_1}. Now two vertices $v_{3,j}$ and $v_{1,j+1}$ have the same distance $j - j_1 + 1$ to the color class 4 and hence have the same color codes, which is a contradiction. For $m \geq 4$, if two consecutive columns have the same missing colors, then by an argument similar to the above, one can find two vertices with the same color codes, which is impossible.
Now let $m \geq 5$ and suppose on the contrary that there exist two (non consecutive) columns C_j and $C_{j'}$ with the same missing color, say "1". We know that there are no two consecutive columns with the same missing colors and hence each of these two columns contains at least one and at most two full colors. Therefore, since $m \geq 5$, there are two vertices v and v' of the same color in the j-th and j'-th columns of G, respectively, such that v and v' are not adjacent to a vertex colored 1. Thus, v and v' have the same color codes, which is a contradiction. $\square$

Theorem 2. *Let $m \geq 3$ and $n \geq 2$ be two positive integers. Then*

$$\chi_{_L}(K_m \Box P_n) = \begin{cases} m+2 & \text{if } m \leq n-2, \\ m+1 & \text{if } m \geq n-1. \end{cases}$$

Proof. Let $G := K_m \square P_n$. By Observation 1(b), we have $\chi_{_L}(G) \geq m+1$. Now we give a locating $(m+2)$-coloring of G. Let the first column of the corresponding coloring matrix be the column vector $[(m+1)\ 1\ 2\ 3\ ...\ (m-2)\ (m+2)]^T$, and the remaining columns be alternately $[1\ 2\ 3\ ...\ m]^T$ and $[m\ 1\ 2\ 3\ ...\ (m-1)]^T$. Then, no two distinct vertices with the same color have the same distances to both of the color classes $m+1$ and $m+2$. Hence, this is a locating coloring of G. Therefore, $\chi_{_L}(G) = m+1$ or $\chi_{_L}(G) = m+2$.
First we show that if $\chi_{_L}(G) = m+1$, then $m \geq n-1$. Therefore, if $m \leq n-2$, then $\chi_{_L}(G) = m+2$.
Assume that $\chi_{_L}(G) = m+1$ and let C be the corresponding matrix of a locating $(m+1)$-coloring of G. Note that C has m rows and in each column exactly one color is missing. By Lemma 1, no two consecutive columns of C have the same missing color, and hence each column of C contains at least one full color. This implies that G has at most $m+1$ columns, i.e. $n \leq m+1$ as desired.
To complete the proof, we assume $m \geq n-1$ and show that $\chi_{_L}(G) = m+1$. For $m \in \{3, 4\}$, consider two colorings of $K_3 \square P_4$ and $K_4 \square P_5$ with the corresponding matrices A_1 and A_2, respectively, as follows.

$$A_1 = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 1 & 4 & 1 \\ 3 & 2 & 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 & 1 & 5 & 4 \\ 2 & 3 & 5 & 3 & 5 \\ 3 & 1 & 2 & 4 & 2 \\ 4 & 2 & 4 & 1 & 3 \end{bmatrix}.$$

Note that in these colorings distinct columns have distinct missing colors and hence, two vertices with the same color have distinct color codes except when both of them are colorful. There are exactly $m+1$ colorful vertices (with distinct colors). Thus, these colorings are locating. Also note that removing columns from the end will not create new full colors in the remaining matrices. Thus, for $m \in \{3, 4\}$ and $n \leq m+1$, we have $\chi_{_L}(K_m \square P_n) = m+1$.
Now let $m \geq 5$. By Lemma 1, in the corresponding matrix of each locating $(m+1)$-coloring of G, if it exists, there are no two distinct columns with the same missing color. In an inductive way, we give a locating $(m+1)$-coloring of G. Equivalently, we fill the columns of an m by n matrix C with entries in $[m+1]$ in such a way that each column contains exactly one full color, distinct columns have distinct full colors, and the missing colors of no two columns are the same. These coloring will be locating since there are no two colorful vertices with the same color and, two non-colorful vertices with the same color are in dif-

ferent columns and are non-adjacent to different color classes. We construct this coloring matrix for $n = m + 1$, then for smaller n one can remove extra columns from the end.

Let $C_1 := [1\ 2\ 3\ ...\ m]^T$ and $C_2 := [(m+1)\ 1\ 2\ 3\ ...\ (m-1)]^T$ be the first and second columns of C, respectively. Now assume that p-th column of C is $C_p = [x_1\ x_2\ x_3\ ...\ x_m]^T$ with the missing color x_{m+1} and, without loss of generality, with the full color x_1, where $\{x_1, x_2, x_3, ..., x_{m+1}\} = \{1, 2, 3, ..., m+1\}$. Next, we fill the column C_{p+1}. Since C_p should contain exactly one full color, the color x_{m+1} should appear in the first row of C_{p+1}.

For each $t \geq 1$, let C_t^F and C_t^M be the singleton sets that contain the full color and the missing color of the column C_t, respectively. If x_1 is not the missing color of one of the previous columns, and x_{m+1} is not the full color of one of the previous columns, then let $C_{p+1} := [x_{m+1}\ x_m\ x_2\ x_3\ x_4\ ...\ x_{m-1}]^T$, which means $C_{p+1}^M = \{x_1\}$ and $C_{p+1}^F = \{x_{m+1}\}$. Otherwise, $x_1 \in \bigcup_{t=1}^p C_t^M$ or $x_{m+1} \in \bigcup_{t=1}^p C_t^F$. If $p+1 < n$, then there are at least two colors not in $\bigcup_{t=1}^p C_t^F$ and there are at least two colors not in $\bigcup_{t=1}^p C_t^M$. Therefore, there exists a color $x_j \notin \bigcup_{t=1}^p C_t^F$, where $x_j \neq x_{m+1}$. Also, there exists a color $x_i \notin \bigcup_{t=1}^p C_t^M$, where $x_i \notin \{x_1, x_j\}$. Choose x_j as the full color, and x_i as the missing color of C_{p+1}. Since $m \geq 5$, it is possible to fill the column C_{p+1} in such a way that its first row is x_{m+1} and its i-th row is x_j. Here after assume that $p+1 = n$, and hence there is only one color not in $\bigcup_{t=1}^p C_t^F$ and only one color not in $\bigcup_{t=1}^p C_t^M$. Following two cases may occur.

Case 1. $[n] \setminus \bigcup_{t=1}^{n-1} C_t^M = \{x_1\}$ and $[n] \setminus \bigcup_{t=1}^{n-1} C_t^F = \{x_i\}$, where $x_i \neq x_{m+1}$.

In this case, we could change some of the previous columns to get the desired coloring. Assume that $C_{n-2}^F = \{x_s\}$ and $C_{n-2}^M = \{x_j\}$. Note that there are no repeated full colors or repeated missing colors, but it may be that $x_s = x_{m+1}$ or $x_i = x_j$.

(a) If $x_s \neq x_{m+1}$ and $x_i = x_j$, then let x_j, x_1 and x_{m+1} be the missing colors of C_{n-2}, C_{n-1} and C_n, respectively. Also let x_1, x_j and x_s be the full colors of C_{n-2}, C_{n-1} and C_n, respectively. Now fill C_{n-2} such that x_1 in C_{n-2} and x_j in C_{n-3} are in the same row, and then fill C_{n-1} such that $x_i = x_j$ in C_{n-1} and x_1 in C_{n-2} are in the same row. Finally, fill C_n such that x_1 in C_n and x_i in C_{n-1} are in the same row, and also x_s in C_n and x_{m+1} in C_{n-1} are in the same row.

(b) If $x_s = x_{m+1}$ and $x_i \neq x_j$, then let $C^M_{n-2} = \{x_j\}, C^M_{n-1} = \{x_1\}, C^M_n = \{x_s\}$ and $C^F_{n-2} = \{x_i\}, C^F_{n-1} = \{x_s\}, C^F_n = \{x_1\}$. Now fill C_{n-2} such that x_i in C_{n-2} and x_j in C_{n-3} are in the same row. Then fill C_{n-1} such that x_j in C_{n-1} and x_i in C_{n-2} are in the same row, and also x_s in C_{n-1} and x_1 in C_{n-2} are in the same row. Finally, fill C_n such that x_1 in C_n and x_s in C_{n-1} are in the same row.

(c) If $x_s \neq x_{m+1}$ and $x_i \neq x_j$, then let $C^M_{n-2} = \{x_j\}, C^M_{n-1} = \{x_{m+1}\}, C^M_n = \{x_1\}$ and $C^F_{n-2} = \{x_1\}, C^F_{n-1} = \{x_s\}, C^F_n = \{x_i\}$. Now fill C_{n-2} such that x_1 in C_{n-2} and x_j in C_{n-3} are in the same row. Then fill C_{n-1} such that x_j in C_{n-1} and x_1 in C_{n-2} are in the same row, and also x_s in C_{n-1} and x_{m+1} in C_{n-2} are in the same row. Finally, fill C_n such that x_{m+1} in C_n and x_s in C_{n-1} are in the same row, and also x_i in C_n and x_1 in C_{n-1} are in the same row.

(d) If $x_s = x_{m+1}$ and $x_i = x_j$, then we should change the column C_{n-3}. Assume that $C^F_{n-3} = \{x_l\}$ and $C^M_{n-3} = \{x_k\}$. Note that $x_l \notin \{x_1, x_i, x_{m+1}\}$ and $x_k \notin \{x_1, x_i, x_{m+1}\}$, since there are no repeated full colors or repeated missing colors. For the desired coloring, let $C^M_{n-3} = \{x_k\}, C^M_{n-2} = \{x_i\}, C^M_{n-1} = \{x_{m+1}\}, C^M_n = \{x_1\}$ and $C^F_{n-3} = \{x_1\}, C^F_{n-2} = \{x_{m+1}\}, C^F_{n-1} = \{x_i\}, C^F_n = \{x_l\}$. Now fill C_{n-3} such that x_1 in C_{n-3} and x_k in C_{n-4} are in the same row. Then fill C_{n-2} such that x_k in C_{n-2} and x_1 in C_{n-3} are in the same row, and also x_{m+1} in C_{n-2} and x_i in C_{n-3} are in the same row. Next, fill C_{n-1} such that x_i in C_{n-1} and x_{m+1} in C_{n-2} are in the same row. Finally, fill C_n such that x_{m+1} in C_n and x_i in C_{n-1} are in the same row, and also x_l in C_n and x_1 in C_{n-1} are in the same row.

Case 2. $[n] \setminus \bigcup_{t=1}^{n-1} C^M_t = \{x_i\}$ and $[n] \setminus \bigcup_{t=1}^{n-1} C^F_t = \{x_{m+1}\}$, where $x_i \neq x_1$.

We should change C_{n-2} to get the desired coloring. Assume that $C^F_{n-2} = \{x_s\}$ and $C^M_{n-2} = \{x_j\}$. Note that $x_j \notin \{x_1, x_i, x_{m+1}\}$ and $x_s \notin \{x_1, x_{m+1}\}$, since there are no repeated full colors or repeated missing colors. But it may be that $x_i = x_s$.

(a) If $x_i \neq x_s$, then let $C^M_{n-2} = \{x_j\}, C^M_{n-1} = \{x_i\}, C^M_n = \{x_{m+1}\}$ and $C^F_{n-2} = \{x_{m+1}\}, C^F_{n-1} = \{x_1\}, C^F_n = \{x_s\}$. Now fill C_{n-2} such that x_{m+1} in C_{n-2} and x_j in C_{n-3} are in the same row. Then fill C_{n-1} such that x_j in C_{n-1} and x_{m+1} in C_{n-2} are in the same row, and also x_1 in C_{n-1} and x_i in C_{n-2} are in the same row. Finally, fill C_n such that x_i in C_n and x_1 in C_{n-1} are in the same row, and x_s in C_n and x_{m+1} in C_{n-1} are in the same row.

(b) If $x_i = x_s$, then let $C^M_{n-2} = \{x_j\}, C^M_{n-1} = \{x_{m+1}\}, C^M_n = \{x_s\}$ and $C^F_{n-2} = \{x_1\}, C^F_{n-1} = \{x_s\}, C^F_n = \{x_{m+1}\}$. Now fill C_{n-2} such that x_1 in C_{n-2} and x_j in C_{n-3} are in the same row. Then fill C_{n-1} such that x_j in C_{n-1} and x_1 in C_{n-2} are in the same row, and also x_s in C_{n-1} and x_{m+1} in C_{n-2} are in the same row. Finally, fill C_n such that x_{m+1} in C_n and x_s in C_{n-1} are in the same row.

Note that since $m \geq 5$, in all of the previous steps it is possible to fill each column in the desired manner. $\square$

3 The locating chromatic number of $K_m \square K_n$

In this section, we study the cartesian product of complete graphs. Let $G := K_m \square K_n$. Vertices of G can be considered as the entries of an m by n matrix, such that the induced subgraph on the vertices of each column is isomorphic to K_m and the induced subgraph on each row is isomorphic to K_n. Let $v_{i,j}$ be the vertex of G in the i-th row and j-th column. Each coloring of G can also be considered as an m by n matrix.

Lemma 2. *Let $m \geq 2$ and $n \geq 3$ be two positive integers, where $m \leq n$. If there exists a locating $(n+1)$-coloring of $G := K_m \square K_n$, then let C be its corresponding coloring matrix. Then different rows of C have different missing colors.*

Proof. Each row has one missing color. Since each color appears in at least one row, the missing color of each row appears in some other rows. Hence, each row contains some full colors.

Suppose on the contrary, and without loss of generality, that first and second rows have the same missing color, say "$n+1$". For each $i \in [n]$, there are two vertices in the first and second rows of G with color i. They have neighbors in all of the color classes $[n] \setminus \{i\}$. Since the coloring is locating, the color $n+1$ should appear in exactly one of the columns corresponding to these two vertices. This holds for each $i \in [n]$. Hence, the color $n+1$ should appear in exactly half of the columns of C. This also implies that n is an even integer. Thus, in each row with the missing color $n+1$, half of the colors are full. Particularly, half of the colors $1, 2, ..., n$ are full in the first row, and the remaining are full in the second row. Since repeated full colors are not allowed, the color $n+1$ must appear in the third row. The missing color of the third row appears in the first and second rows, in two different columns. Hence, the third row contains at least two full colors. This implies that there are at least $n+2$ full colors in $n+1$ color classes, which is a contradiction. $\square$

Note that $\chi_{_L}(K_2 \square K_2) = \chi_{_L}(C_4) = 4$. In general we have the following result.

Theorem 3. *For two positive integers $m \geq 2$ and $n \geq 3$, where $m \leq n$, let*

$$m_0 := \max\{k \mid k \in \mathbb{N},\ k(k-1) - 1 \leq n\}.$$

(a) *If $m \leq m_0 - 1$, then $\chi_{_L}(K_m \square K_n) = n+1$,*

(b) *If $m_0 + 1 \leq m \leq \frac{n}{2}$, then $\chi_{_L}(K_m \square K_n) = n+2$.*

Proof. Let $G := K_m \square K_n$. By Observation 1(b), we have $\chi_{_L}(G) \geq n+1$.
If $m = 2$, then the following matrix provides a locating $(n+1)$-coloring of G with the color set $[n+1]$.

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & 1 & 2 & \dots & n-1 \end{bmatrix}$$

Let $m = 3$. If $n = 3$, then it is not hard to see that there exists no locating 4-coloring of $K_3 \square K_3$. The following matrix A_1 gives a locating 5-coloring of $K_3 \square K_3$. Hence $\chi_{_L}(K_3 \square K_3) = 5$.
If $n = 4$ and $\chi_{_L}(K_3 \square K_4) = n+1 = 5$, then by Lemma 2 different rows have different missing colors. Hence, each row contains two full colors, which is impossible since there are only five color classes. The following matrix A_2 gives a locating 6-coloring of $K_3 \square K_4$, and hence $\chi_{_L}(K_3 \square K_4) = 6$. If $n = 5$, then the matrix A_3 gives a locating 6-coloring of $K_3 \square K_5$, and hence $\chi_{_L}(K_3 \square K_5) = 6$.

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 6 & 5 & 1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 5 & 3 & 4 & 2 \\ 6 & 1 & 5 & 2 & 4 \\ 3 & 4 & 2 & 5 & 6 \end{bmatrix}$$

Finally, the following matrix gives a locating $(n+1)$-coloring of $n \geq 6$, and hence $\chi_{_L}(K_3 \square K_n) = n+1$.

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ n+1 & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 2 & 3 & 4 & \dots & n-1 & n & n+1 \end{bmatrix}$$

Here after let $m \geq 4$. First assume there exists a locating $(n+1)$-coloring c of G with the corresponding coloring matrix C. By Lemma 2, missing colors of different rows are different, and the missing color of each row appears in all of the other $m-1$ rows. Hence, each row of C contains exactly $m-1$ full colors, and there are exactly $m(m-1)$ full colors in C. Since there are $n+1$ color classes, we should have $m(m-1) \leq n+1$. Therefore, if $m > m_0$, then $\chi_{_L}(G) \geq n+2$. Now we provide a locating $(n+2)$-coloring of G, when $m_0 + 1 \leq m \leq \frac{n}{2}$. Let $c : V(G) \longrightarrow [n+2]$ be a function, where $c(v_{i,j}) := (i-1)n + j \pmod{(n+2)}$. Since $m \leq \frac{n}{2}$, by a simple calculation, it can be seen that c is a proper coloring. For each i, $1 \leq i \leq m$, the i-th row of the corresponding coloring matrix has two missing colors $in+1$ and $in+2$, modulo $n+2$. Moreover, since $m \leq \frac{n}{2}$, two colors $in+1$ and $in+2$ can not appear in the same column. This means that there exists no full color. In other words, each vertex has at least one component that is 2 in its color code. Since each color is missed in exactly one row, every two vertices with the same color are in different rows and have different color codes. Consequently, c is a locating $(n+2)$-coloring. Hence, $\chi_{_L}(G) = n+2$, when $m_0 < m \leq \frac{n}{2}$.

To complete the proof, we show that $\chi_{_L}(G) = n+1$, when $m < m_0$. In an inductive way, we provide a locating $(n+1)$-coloring of $K_m \square K_n$. For this purpose, we construct an m by n (coloring) matrix on the set $[n+1]$ with the following properties.

(a) The entries of each row, and each column are different.

(b) Different rows have different missing colors.

(c) There exist no repeated full colors.

(d) All of the missing colors are also full.

Note that the property (a) indicates that the given coloring is proper. Also, the properties (b) and (c) guarantee that the coloring is a locating coloring, since two non-colorful vertices with the same color are in different rows and are non-adjacent to different color classes. The property (d) is needed for the proof by induction. If $m = 4$, then $m_0 \geq 5$ and $n \geq 19$. Consider the following coloring matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots & i & \dots & n-5 & n-4 & n-3 & n-2 & n-1 & n \\ n+1 & 1 & 2 & 3 & 4 & \dots & i-1 & \dots & n-6 & n-5 & n-4 & n-3 & n-2 & n-1 \\ n & n+1 & 1 & 2 & 3 & \dots & i-2 & \dots & n-7 & n-6 & n-5 & n-4 & n-3 & n-2 \\ 4 & 5 & 6 & 7 & 8 & \dots & i+3 & \dots & n-2 & n-1 & n & 2 & 3 & n+1 \end{bmatrix}$$

In this coloring, the missing colors are $n+1, n, n-1, 1$, and the full colors are $1, 2, n; n+1, n-4, n-1; n-6, n-3, n-2; 4, 5, 6$. Thus, there are no repeated full colors, whenever $n \geq 13$, and properties (a) to (d) hold.

Now suppose that the i-th row, $4 < i \leq m-1$, is completed such that the properties (a) to (d) hold for the constructed $i \times n$ matrix. Next, we complete the $(i+1)$-th row. Without loss of generality and by permuting the rows or symbols if it is necessary, assume that the missing colors in the first i rows are $1, 2, ..., i$. Each of the first i rows contains $i-1$ full colors, since its missing color appears in other rows. Thus, there are $i(i-1)$ full colors and i missing colors. Choose a full color j, $j > i$. We want to fill the $(i+1)$-th row with colors in the set $[n+1] \setminus \{j\}$ in such a way that the constructed $(i+1) \times n$ matrix satisfies the properties (a) to (d).

By completing the $(i+1)$-th row, $2i$ new colorful vertices appear, i of them will be in the first i rows (one vertex in each row) by inserting the colors $1, 2, ..., i$ in the $(i+1)$-th row, and i of them will be in the $(i+1)$-th row corresponding to the columns in which j occurs in the previous rows.

Let $1 \leq k \leq i$. The color k should be inserted in a suitable column in the $(i+1)$-th row, in such a way that it creates a new full color in the k-th row. Since the colors $1, 2, ..., i$ are full, to preserve the property (c), these colors shouldn't be inserted in the columns of the $(i+1)$-th row in which j occurs in the previous rows. There are $n-i$ columns not containing j. On the other hand, $i(i-1)$ full colors of the first i rows appear in the k-th row and inserting k in their columns causes repeated full colors. Also, one of these full colors is j and each column containing k contains at least one full color. Thus, there are at least $(n-i)-i(i-1)+1 = n-i^2+1$ possible columns in the $(i+1)$-th row for inserting the color k. Assume that the color 1 is inserted in a suitable column. After inserting 1, one new full color is created in the first row and one of the feasible columns of the $(i+1)$-th row is occupied by 1. Hence, for inserting 2 there are at least $(n-i^2+1)-2$ possible columns, and finally for inserting i there are at least $(n-i^2+1)-2(i-1)$ possible columns. Note that $(n-i^2+1)-2(i-1) \geq 1$, since $i < m < m_0$ and $m_0(m_0-1)-1 \leq n$. Thus, inserting the colors $1, 2, ..., i$ is possible as desired.

Inserting each color in the columns in which j occurs in the previous rows, will make that color full. There are i columns containing the color j. Since

$$
\begin{aligned}
(n+1) - i(i-1) - i &= n + 1 - i^2 \\
&\geq m_0(m_0-1) - i^2 \\
&\geq m_0(m_0-1) - (m_0-2)^2 \\
&= 3(m_0-2) + 2 \\
&\geq 3i + 2,
\end{aligned}
$$

there are at least $3i+2$ non full colors. Therefore, it is possible to insert i non full colors in the $(i+1)$-th row, and in the columns in which j occurs in the first i rows, preserving the property (a).
Now it remains to insert the remaining $n-2i$ colors, say $c_1, c_2, ..., c_{n-2i}$, in the remaining $n-2i$ columns, say $C_1, C_2, ..., C_{n-2i}$, preserving the property (a). Let $H := (X, Y)$ be the bipartite graph with partite sets $X := \{C_1, C_2, ..., C_{n-2i}\}$ and $Y := \{c_1, c_2, ..., c_{n-2i}\}$ such that $C_s c_r \in E(H)$, whenever the color c_r is not occurred in the column C_s. Each color c_r is in i rows and each column C_s contains i colors. Thus, each vertex in H has degree at least $n-3i$. Let $\emptyset \subset S \subset X$. Since $S \neq \emptyset$, $|N(S)| \geq n-3i$. If $|N(S)| < |S|$, then $n-3i < |S|$. Thus, $N(S) \neq Y$ and $|X \setminus S| \leq i-1$. Let $y \in Y \setminus N(S)$ and hence,

$$n - 3i \leq |N(y)| \leq |X \setminus S| \leq i - 1.$$

Thus, $n \leq 4i - 1$ and

$$m_0(m_0 - 1) - 1 \leq 4i - 1 \leq 4(m_0 - 1) - 1.$$

This implies that $m_0 \leq 4$ which is a contradiction, since $4 \leq m < m_0$. Therefore, the Hall's condition holds (Theorem 3.1.11 [10]) and hence H has a perfect matching. Consequently, we obtain a desired coloring by filling the remaining entries according to this assignment. $\square$

4 Some open problems

Note that every proper coloring of $K_m \square K_n$ is equivalent to an m by n Latin rectangle. Moreover, a locating coloring of $K_m \square K_n$ is equivalent to an m by n Latin rectangle in which, for every two cells containing the same symbol, there is a symbol that appears only in the row or column of one of them.
In what follows we present some open problems related to the obtained results.

Note that for each given number n, $m_0 = \max\{k \mid k \in \mathbb{N},\ k(k-1) - 1 \leq n\}$ is a number close to $\sqrt{n}$. If $m = m_0 \geq 4$ and $(m_0(m_0-1)-1) + (m_0-2) \leq n < (m_0+1)m_0 - 1$, then for each i with $i < m_0$, we have

$$(n - i^2 + 1) - 2(i - 1) \geq 1, \quad (n + 1) - i(i - 1) - i \geq 2m_0 - 3,$$

and $n \nleq 4i - 1$. Thus, by following the proof of the Theorem 3, we can obtain a locating $(n+1)$-coloring of $K_{m_0} \square K_n$ and hence, $\chi_L(K_{m_0} \square K_n) = n+1$. Therefore, if $m = m_0$, then the remaining cases for n to investigate $\chi_L(K_{m_0} \square K_n)$ are

$$m_0(m_0-1)-1 \leq n \leq (m_0(m_0-1)-1)+(m_0-3).$$

Verifying small cases, encourage us to give the following conjecture.

Conjecture 1. *$\chi_L(K_{m_0} \square K_n) = n+1$, where $n \geq 3$ and $m_0 = \max\{k \mid k \in \mathbb{N},\ k(k-1)-1 \leq n\}$.*

By a long detailed argument, we can prove that $\chi_L(K_m \square K_n) = n+2$ for $\frac{n}{2} < m \leq \lceil \frac{n+3}{2} \rceil$. For the remaining cases we provide the following conjecture.

Conjecture 2. *If $\lceil \frac{n+3}{2} \rceil < m \leq n$, then $\chi_L(K_m \square K_n) = n+3$.*

The similarity of the structures of $K_m \square C_n$ and $K_m \square P_n$ is a motivation for the following conjecture.

Conjecture 3. *If m and n are sufficiently large, then*

$$\chi_L(K_m \square C_n) = \chi_L(K_m \square P_n).$$

It seems that graphs with bigger diameter have smaller locating chromatic number. Hence, the obtained results suggest the following conjecture.

Conjecture 4. *For every two connected graphs G and H,*

$$\chi_L(G \square H) \leq \max\{\chi_L(G), \chi_L(H)\} + 3.$$

References

[1] Asmiati, H. Assiyatun, and E.T. Baskoro, Locating-chromatic number of amalgamation of stars, *ITB J. Sci.* **43, A** (2011) 1-8.

[2] A. Behtoei and B. Omoomi, On the locating chromatic number of Kneser graphs, *to appear in Discrete Appl. Math.*

[3] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, and P. Zhang, The locating-chromatic number of a graph, *Bull. Inst. Combin. Appl.* **36** (2002) 89-101.

[4] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, and P. Zhang, Graphs of order n with locating-chromatic number $n-1$, *Discrete Math.* **no. 13, 269** (2003) 65-79.

[5] G. Chartrand, F. Okamoto, and P. Zhang, The metric chromatic number of a graph, *Australasian Journal of Combinatorics* **44** (2009) 273-286.

[6] G. Chartrand, V. Saenpholphat, and P. Zhang, Resolving edge colorings in graphs, *Ars Combin.* **74** (2005) 33-47.

[7] G. Chartrand, E. Salehi, and P. Zhang, The partition dimension of a graph, *Aequationes Math.* **no. 1-2, 59** (2000) 45-54.

[8] F. Harary, and R.A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976) 191-195.

[9] V. Saenpholphat and P. Zhang, Conditional resolvability in graphs: A survey, *Int. J. Math. Sci.* **37-40** (2004) 1997-2017.

[10] D. B. West, Introduction to graph theory, *Prentice Hall Inc., Upper Saddle River, NJ, second edition,* (2001).