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On the locating chromatic number of Kneser graphs

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ABSTRACT

Let c be a proper k-coloring of a connected graph G and $\Pi = (C_1, C_2, \ldots, C_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple $c_\Pi(v) := (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x)|x \in C_i\}$, $1 \le i \le k$. If distinct vertices have distinct color codes, then c is called a locating coloring. The minimum number of colors needed in a locating coloring of G is the locating chromatic number of G, denoted by $\chi_L(G)$. In this paper, we study the locating chromatic number of Kneser graphs. First, among some other results, we show that $\chi_L(KG(n, 2)) = n - 1$ for all $n \ge 5$. Then, we prove that $\chi_L(KG(n, k)) \le n - 1$, when $n \ge k^2$. Moreover, we present some bounds for the locating chromatic number of odd graphs.

1. Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of G is a function c defined from $V(G)$ onto a set of colors $C = \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \le i \le k$, the set $c^{-1}(i)$ is a nonempty independent set of vertices which is called the color class i. The minimum cardinality k for which G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$.

For a connected graph G, the distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them, and for a subset S of $V(G)$, the distance between u and S is given by $d(u, S) := \min\{d(u, x) \mid x \in S\}$. A set $W \subseteq V(G)$ is called a *resolving set*, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \ne d(v, w)$, see [9,16]. A resolving set with the minimum cardinality is called a *metric basis* and its cardinality is called the *metric dimension* of G, denoted by $\dim_M(G)$.

Definition 1 (*[2]*). Let c be a proper k-coloring of a connected graph G and $\Pi = (C_1, C_2, \ldots, C_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple

$$c_\Pi(v) := (d(v, C_1), d(v, C_2), \ldots, d(v, C_k)).$$

If distinct vertices of G have distinct color codes, then c is called a locating coloring of G. The locating chromatic number, $\chi_L(G)$, is the minimum number of colors in a locating coloring of G.

The concept of locating coloring was first introduced by Chartrand et al. in [2] and studied further in [1,3]. This concept has been called with the other names such as *resolving coloring* and *independent resolving partition*, see [13]. Note that, since every locating coloring is a proper coloring, $\chi(G) \le \chi_L(G)$. For more results on the subject and related subjects, one can see [1–6,13,14].

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We have the following two theorems for the relation between the locating chromatic number of a graph with its diameter, metric dimension and chromatic number.

Theorem A ([2]). *If G is a connected graph with diameter d and locating chromatic number l, then* $|V(G)| \leq ld^{l-1}$.

Theorem B ([2]). *For each connected graph G with at least three vertices,* $\chi_L(G) \leq \chi(G) + \dim_M(G)$.

Hereafter, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the collection of all k-subsets of the set $[n]$ by $\binom{[n]}{k}$. Let n and k be two positive integers. The Kneser graph with parameters n and k, $n \geq 2k$, denoted by $KG(n, k)$, is the graph with vertex set $\binom{[n]}{k}$ such that two vertices are adjacent if and only if the corresponding subsets are disjoint. Let $k \geq 3$. Kneser graph $KG(2k, k)$ is a matching and the smallest positive integer n for which $KG(n, k)$ is connected, is $n = 2k + 1$. Kneser graphs $KG(2k + 1, k)$, $k \geq 3$, are known as the odd graphs. The distance between two vertices in Kneser graph and the diameter of this graph are investigated in [18]. We summarize these results in the following theorem.

Theorem C ([18]). *Let $A, B \in \binom{[n]}{k}$ be two different vertices of Kneser graph $KG(n, k)$, where $n \geq 2k + 1$. If $|A \cap B| = s$, then the distance $d(A, B)$ in $KG(n, k)$ is given by*

$$d(A, B) = \min\left\{2\left\lceil \frac{k-s}{n-2k} \right\rceil, 2\left\lceil \frac{s}{n-2k} \right\rceil + 1\right\}.$$

Moreover, the diameter of $KG(n, k)$ is $\lceil \frac{k-1}{n-2k} \rceil + 1$.

Kneser graphs have many interesting properties and have been the subject of many researches. It was conjectured by Kneser in 1955 [11] and proved by Lovász in 1978 [12] that $\chi(KG(n, k)) = n - 2k + 2$. Since then, several types of colorings of Kneser graphs have been considered. For example, the circular chromatic number, the b-chromatic number and the multichromatic number of Kneser graphs were investigated in [8,10,17], respectively.

In this paper, we study the locating chromatic number of Kneser graphs. In the next section, among some other results, we show that $\chi_L(KG(n, 2)) = n - 1$ for $n \geq 5$. Then, we prove that $\chi_L(KG(n, k)) \leq n - 1$ when $n \geq k^2$. For the case $k = 3$, we show that this inequality holds for every positive integer $n \geq 7$. In the last section, we provide a lower bound for the locating chromatic number, an upper bound for the metric dimension and accordingly for the locating chromatic number of odd graphs. Through the paper, for convenience, we denote the vertex $\{i_1, i_2, \ldots, i_k\}$ in $KG(n, k)$ by $i_1 i_2 \ldots i_k$.

2. The locating chromatic number of $KG(n, 2)$

If A is an independent set in $KG(n, 2)$, then either all vertices in A have a common element of $[n]$, say a, or $A = \{ab, ac, bc\}$ for some $a, b, c \in [n]$. Since each vertex ij in $KG(n, 2)$ corresponds to the edge ij in K_n, an independent set in $KG(n, 2)$ corresponds to a star subgraph or a triangle subgraph in the complete graph K_n. From now on, we call an independent set in $KG(n, 2)$ of the first form *starlike* with center a, and of the second form *triangular*. Since every proper coloring is a partition of vertices into independent sets, it is easy to see that every proper coloring of the Kneser graph $KG(n, 2)$ is equivalent to an edge decomposition of the complete graph K_n into star and triangle subgraphs.

In order to study the locating chromatic number of $KG(n, 2)$, we need the following theorem. A *biclique partition* of a graph G is a partition of the edge set of G into complete bipartite graphs. Since a single edge can form a biclique, every graph has a biclique partition. The *biclique partition number* $\mathrm{bp}(G)$ of G is the smallest number of bicliques that partition G. Since the complete graph K_n can be partitioned into $n - 1$ stars, $\mathrm{bp}(K_n) \leq n - 1$. In fact, we have the following famous theorem.

Theorem D ([7]). *The biclique partition number of the complete graph K_n is $n - 1$.*

Consider the Kneser graph $KG(n, k)$, $n > 2k$. Let $n = 2k + d, d \geq 1$. There is a proper coloring of $KG(n, k)$ with $\chi(KG(n, k)) = d + 2$ colors as follows. For $i = 1, 2, \ldots, d + 1$, let C_i consist of all k-subsets of $[n]$ which contain i as the smallest element. The remaining k-subsets are contained in the set $\{d + 2, d + 3, \ldots, d + 2k\}$, which has only $2k - 1$ elements. Hence, they all intersect (are non adjacent). Thus, we can use color $d + 2$ for all of them.

For the case $k = 2$, $\chi(KG(n, 2)) = n - 2$ and the latter color class in the above proper coloring is of triangular form. In this section, we first show that all proper $(n - 2)$-colorings of $KG(n, 2)$ are similar to the proper coloring given above. Next, we determine the exact value of the locating chromatic number of $KG(n, 2)$.

Theorem 1. *In every proper $(n - 2)$-coloring of the Kneser graph $KG(n, 2)$, $n \geq 5$, there exists a unique triangular color class. Furthermore, if c is a proper $(n - 2)$-coloring of $KG(n, 2)$, then by renaming the symbols $1, 2, \ldots, n$, if it is necessary, we have the color classes $F_1, F_2, \ldots, F_{n-2}$ with the following properties.*

(a) $F_{n-2} = \{n(n-1), n(n-2), (n-1)(n-2)\}$, *i.e.* F_{n-2} *is triangular;*
(b) *for each i, $1 \leq i \leq n - 3$, F_i is starlike with center i, and* $\{in, i(n-1), i(n-2)\} \subseteq F_i$*;*
(c) *each vertex ij with* $\{i, j\} \bigcap \{n, n-1, n-2\} = \emptyset$, *is either in F_i or F_j.*

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Proof. We will prove the theorem by induction on n. Let $n = 5$ and consider a proper 3-coloring of $KG(n, 2)$ with color classes F_1, F_2 and F_3. Equivalently, we have an edge decomposition of the complete graph K_5 into stars and triangles. By Theorem D, there exists no edge decomposition of K_5 into three star subgraphs. Thus, at least one of the color classes is triangular, say $F_3 = \{34, 35, 45\}$. Now, six vertices 13, 14, 15; 23, 24, 25 should be distributed between two color classes F_1 and F_2. Since F_1 and F_2 are independent sets, the only possibility is, say $\{13, 14, 15\} \subseteq F_1$ and $\{23, 24, 25\} \subseteq F_2$, which means F_1 and F_2 are starlike with centers 1 and 2, respectively. The remaining vertex 12 can be either in F_1 or F_2. Hence, the theorem holds for the induction basis.

Now, let $n \geq 6$ and suppose that c is a proper $(n-2)$-coloring of $KG(n, 2)$. Equivalently, we have an edge decomposition of the complete graph K_n into stars and triangles. Similar to the one above, by Theorem D, we have at least one triangular color class, say $F_{n-2} = \{n(n-1), n(n-2), (n-1)(n-2)\}$. Let

$$X_n := \bigcup_{i=1}^{n-3}\{in\}, \qquad X_{n-1} := \bigcup_{i=1}^{n-3}\{i(n-1)\}, \qquad X_{n-2} := \bigcup_{i=1}^{n-3}\{i(n-2)\}$$

and $X := X_n \cup X_{n-1} \cup X_{n-2}$.

Note that, each X_i is starlike, $|X| = 3(n-3)$, and for each $i, j,\ n-2 \leq i < j \leq n$, the induced subgraph of $KG(n, 2)$ on $X_i \cup X_j$ is a complete bipartite graph with a perfect matching deleted. The vertices in X should be distributed in $n-3$ color classes $F_1, F_2, \ldots, F_{n-3}$. Thus, there exists a color class, say F_1, which contains at least three vertices of X.

Claim. *There exists no $k,\ n-2 \leq k \leq n$, such that $F_1 \cap X \subseteq X_k$.*

Proof of claim. Assume to the contrary, and without loss of generality, that $F_1 \cap X \subseteq X_n$. Since F_1 is an independent set and $|F_1 \cap X| \geq 3$, F_1 is starlike with center n. Now, the $2(n-3) = 2n-6$ vertices of $X_{n-1} \cup X_{n-2}$ should be distributed in the $n-4$ color classes $F_2, F_3, \ldots, F_{n-3}$. Hence, there exists a color class, say F_2, which contains at least three vertices of $X_{n-1} \cup X_{n-2}$. Since each vertex $i(n-1)$ in X_{n-1} is adjacent to all of the vertices in X_{n-2} except the vertex $i(n-2)$, the only possibility is $F_2 \cap (X_{n-1} \cup X_{n-2}) \subseteq X_{n-1}$ or $F_2 \cap (X_{n-1} \cup X_{n-2}) \subseteq X_{n-2}$.

Without loss of generality, we can assume that $F_2 \cap (X_{n-1} \cup X_{n-2}) \subseteq X_{n-1}$. Similarly, this implies that F_2 is starlike with center $n-1$. Now, for each i, $3 \leq i \leq n-3$, let $\bar{F}_i := F_i$ and also let

$$\bar{F}_1 := F_1 \cup \{(n-1)n, (n-2)n\}, \qquad \bar{F}_2 := F_2 \cup \{(n-1)(n-2)\}.$$

Thus, each $\bar{F}_i$ is an independent set in the Kneser graph $KG(n, 2)$. This means that we have a proper $(n-3)$-coloring of $KG(n, 2)$, which is a contradiction. Thus, the claim is proved.

If $|F_1 \cap X_k| \geq 3$ for some $k,\ n-2 \leq k \leq n$, then F_1 is starlike with center k. Thus, $F_1 \cap X \subseteq X_k$, which is impossible by the above claim. If for some $k,\ n-2 \leq k \leq n$, $|F_1 \cap X_k| = 2$, then there exist i and j, $1 \leq i < j \leq n-3$, such that $F_1 \cap X_k = \{ik, jk\}$. Since F_1 contains at least three vertices from X, F_1 is an independent set and every vertex in $X \setminus X_k$ is adjacent to at least one of the vertices ik or jk, we should have $F_1 \cap X \subseteq X_k$, which by claim is impossible. Therefore, $|F_1 \cap X_k| = 1$ for each k, $n-2 \leq k \leq n$.

By renaming the symbols $1, 2, \ldots, n-3$, if it is necessary, assume that $F_1 \cap X_n = \{1n\}$. Since F_1 is an independent set and the vertex $1n$ is adjacent to all of the vertices in $X_{n-1} \cup X_{n-2}$ except $1(n-1)$ and $1(n-2)$, we have

$$F_1 \cap X = \{1n, 1(n-1), 1(n-2)\}.$$

This implies that F_1 is starlike with center 1. Let

$$\bar{F}_1 := \{12, 13, \ldots, 1n\}$$

and for each $2 \leq j \leq n-2$, let $\bar{F}_j := F_j \setminus \bar{F}_1$. Note that, $\bar{F}_1$ is starlike with center 1 and $|\bar{F}_1| = n-1$, $\bar{F}_j \subseteq F_j$ for each $2 \leq j \leq n-3$, and $\bar{F}_{n-2} = F_{n-2}$. Since $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{n-2}$ is a partition of the vertices of $KG(n, 2)$ into independent sets and $\chi(KG(n, 2)) = n-2$, none of the $\bar{F}_j$'s is an empty set. Now, we can consider $\bar{F}_2, \bar{F}_3, \ldots, \bar{F}_{n-2}$ as a proper $(n-3)$-coloring of the Kneser graph $KG(n-1, 2)$ with vertex set $\binom{[n]\setminus\{1\}}{2}$. By the induction hypothesis, exactly one of these color classes is triangular, which is $\bar{F}_{n-2}$, and by renaming $\bar{F}_j$'s, if it is necessary, each $\bar{F}_j$ is a starlike independent set with center j, $2 \leq j \leq n-3$. Moreover, by the induction hypothesis, we have

$$\{jn, j(n-1), j(n-2)\} \subseteq \bar{F}_j \subseteq F_j, \quad 2 \leq j \leq n-3.$$

Thus, each F_j is a starlike independent set with center j. Using the induction hypothesis, it is easy to see that the condition (c) also holds. $\square$

By Theorem 1, every optimal proper coloring of $KG(n, 2)$ has a unique triangular color class. This set uses exactly three symbols from the set $[n]$ which can be chosen in $\binom{n}{3}$ different ways. If we choose this triangular set, then the colors of $3(n-3)$ vertices will be determined. Now, the remaining $\binom{n}{2} - 3 - 3(n-3)$ vertices should be distributed among $n-3$ starlike color classes and each of them has two choices for their color. Accordingly, we have the following corollary.

Corollary 1. *The number of different optimal proper colorings of the Kneser graph $KG(n, 2)$ is* $\binom{n}{3} 2^{\binom{n}{2}-3(n-2)}$.

Now, we are ready to determine the exact value of the locating chromatic number of $KG(n, 2)$.

Theorem 2. *For all positive integers $n \geq 5$, we have $\chi_L(KG(n, 2)) = n - 1$.*

Proof. By Theorem C, the diameter of $KG(n, 2)$ is two. First, we show that $\chi_L(KG(n, 2)) \geq n - 1$. Since

$$n - 2 = \chi(KG(n, 2)) \leq \chi_L(KG(n, 2)),$$

it is sufficient to show that no proper $(n-2)$-coloring of $KG(n, 2)$ is a locating coloring. Let c be a proper $(n-2)$-coloring of $KG(n, 2)$. By the same notations and assumptions as in Theorem 1, assume that $F_1, F_2, \ldots, F_{n-2}$ are its color classes and let $\prod = (F_1, \ldots, F_{n-2})$ be an ordered partition of the vertex set into the resulting color classes. Now, since $d(1n, (n-1)(n-2)) = d(1(n-1), (n-2)n) = 1$, and for each j, $2 \leq j \leq n-3$, $d(1n, j(n-1)) = d(1(n-1), j(n-2)) = 1$, we have $c_\Pi(1n) = c_\Pi(1(n-1)) = (0, 1, 1, \ldots, 1)$. Thus, c is not a locating coloring.

Now, to complete the proof, we provide a locating $(n-1)$-coloring of $KG(n, 2)$. Let

$$C_1 := \{1j \mid 2 \leq j \leq n-2\} \cup \{1n\}, \qquad C_{n-1} := \{1(n-1), (n-1)n\},$$

and for each i, $2 \leq i \leq n-2$, let $C_i := \{ij \mid i < j \leq n\}$. Note that, each C_i, $1 \leq i \leq n-1$, is starlike with center i and $C_1, C_2, \ldots, C_{n-1}$ is a partition of the vertex set. Therefore, by assigning color i to the vertices in C_i, we get a proper $(n-1)$-coloring of $KG(n, 2)$. Let us show that this is a locating coloring. Let $\prod = (C_1, \ldots, C_{n-1})$ be an ordered partition of the vertex set into the resulting color classes. It is sufficient to show that distinct vertices of each color class have distinct color codes. Since $d(1(n-1), C_1) = 2$ and $d((n-1)n, C_1) = 1$, this holds for the color class C_{n-1}. This also holds for the color class C_{n-2}, since

$$d((n-2)(n-1), C_{n-1}) = 2, \qquad d((n-2)n, C_{n-1}) = 1.$$

Now, consider color class C_i, $1 \leq i \leq n-3$, and let ij and ik be two distinct vertices in C_i with $j < k$. We know that $d(ij, C_j) = 2$. If $k = n$ and $j < n-1$, then $d(ik, j(j+1)) = 1$, where $j(j+1) \in C_j$. If $k = n$ and $j = n-1$, then $i \neq 1$, and $d(ik, 1(n-1)) = 1$, where $1(n-1) \in C_j$. If $k \neq n$, then $d(ik, jn) = 1$, where $jn \in C_j$. Therefore, in all cases we have $d(ik, C_j) = 1$, which implies $c_\Pi(ij) \neq c_\Pi(ik)$. □

3. The locating chromatic number of $KG(n, k)$

In this section, we give an upper bound for the locating chromatic number of some class of Kneser graphs. Note that, an independent set in $KG(n, k)$ is called starlike with center j, whenever j is the unique common symbol of every two vertices in it.

Theorem 3. *Let n, k be two positive integers, where $k \geq 3$. If $n \geq k^2$, then*

$$\chi_L(KG(n, k)) \leq n - 1.$$

Proof. Since $n \geq k^2$ and $k \geq 3$, by Theorem C, the diameter of $KG(n, k)$ is two. To prove the theorem, it is enough to provide a locating $(n-1)$-coloring for $KG(n, k)$. For this purpose, we first construct a family $\{F_2, F_3, \ldots, F_n\}$ of the subsets of the vertex set $\binom{[n]}{k}$ with the following properties.

(a) Each F_j is starlike with center j which contains exactly $k+1$ vertices, and for each two distinct vertices $X, Y \in F_j$, $X \cap Y = \{j\}$.
(b) Each F_j uses exactly $1 + (k-1)(k+1) = k^2$ symbols of the set $[n]$, i.e. there exists a k^2-element subset B of $[n]$ such that each vertex in F_j is a subset of B.
(c) For each $2 \leq r < s \leq n$, $F_r \cap F_s = \emptyset$.
(d) If $2 \leq j \leq n$, then each vertex $A \in \binom{[n]}{k} \setminus F_j$ with $j \notin A$, is adjacent to at least one vertex in F_j and $d(A, F_j) = 1$.

If $k = 3$, then we define the family $\{F_2, F_3, \ldots, F_n\}$ as follows.

$F_2 := \{259, 268, 247, 213\}$, $F_3 := \{324, 319, 358, 367\}$, $F_4 := \{413, 456, 479, 428\}$, $F_5 := \{512, 534, 567, 589\}$, $F_6 := \{612, 634, 658, 679\}$, $F_7 := \{712, 734, 759, 768\}$, $F_8 := \{812, 834, 857, 869\}$, and for each l, $9 \leq l \leq n$, let $F_l := \{l12, l34, l56, l78\}$. It is easy to check that the properties (a) to (d) hold for this family.

If $k \geq 4$, then we construct the family $\{F_2, F_3, \ldots, F_n\}$ as follows.

Let $A_{2,1}, A_{2,2}, \ldots, A_{2,k+1}$ be a partition of the set $[k^2] \setminus \{2\}$ into $k+1$ subsets, each of size $k-1$. Note that this can be done in $\frac{(k^2-1)!}{((k-1)!)^{k+1}\,(k+1)!}$ different ways which is strictly greater than one, since $k > 2$. Now, let

$$F_2 := \bigcup_{j=1}^{k+1} \{A_{2,j} \cup \{2\}\}.$$

In fact, F_2 is an independent subset of the vertex set $\binom{[n]}{k}$ of size $k+1$ which is starlike with center 2. Each vertex in F_2 is a subset of $[k^2]$ and $X \cap Y = \{2\}$ for each two distinct vertices $X, Y \in F_2$. If $A \in \binom{[n]}{k} \setminus F_2$ is a vertex with $2 \notin A$, then it is adjacent to at least one vertex in F_2, thus $d(A, F_2) = 1$. In an inductive way, suppose that $F_2, F_3, \ldots, F_i$, $i < k^2$, are constructed in such a way that each $F_j \subseteq \binom{[k^2]}{k}$, $2 \le j \le i$, is starlike of size $k+1$ with center j, $X \cap Y = \{j\}$ for each two distinct vertices $X, Y \in F_j$, and $F_r \bigcap F_s = \emptyset$ whenever $r \neq s$. Then, we construct F_{i+1} as follows.

For each j, $2 \le j \le i$, let V_j be the unique vertex in F_j which contains the element $i+1$, and let $U_j := V_j \setminus \{i+1\}$. The number of different partitions of the set $[k^2] \setminus \{i+1\}$ into $k+1$ subsets of size $k-1$ in which U_j appears as a partition part of them, is equal to the number of different partitions of the set $[k^2] \setminus (U_j \cup \{i+1\})$ into k subsets of size $k-1$, which is $\frac{(k^2-k)!}{((k-1)!)^k\, k!}$. This implies that, at most

$$(i-1)\frac{(k^2-k)!}{((k-1)!)^k\, k!}$$

different partitions of the set $[k^2] \setminus \{i+1\}$ into $k+1$ subsets of size $k-1$, have some U_j as a partition part. Now we consider the ratio between the number of all possible partitions and the number of partitions containing some U_j as a partition part. Since $k \ge 4$,

$$\begin{aligned}\frac{\frac{(k^2-1)!}{((k-1)!)^{k+1}\,(k+1)!}}{\frac{(k^2-2)(k^2-k)!}{((k-1)!)^k\, k!}} &= \frac{(k^2-3)(k^2-4)\cdots(k^2-k+1)}{(k-2)!}\\ &= \frac{1}{(k-2)} \times \frac{(k^2-3)}{(k-3)} \times \frac{(k^2-4)}{(k-4)} \times \cdots \times \frac{(k^2-(k-1))}{(k-(k-1))} > 1,\end{aligned}$$

and this inequality using the inequality $i-1 \le k^2-2$, implies that

$$\frac{(k^2-1)!}{((k-1)!)^{k+1}(k+1)!} - (i-1)\frac{(k^2-k)!}{((k-1)!)^k\, k!} > 1.$$

Therefore, there exists a partition $A_{i+1,1}, A_{i+1,2}, \ldots, A_{i+1,k+1}$ of the $(k-1)$-subsets of $[k^2] \setminus \{i+1\}$ such that none of them is equal to some U_j, $1 \le j \le i$. Now, let

$$F_{i+1} := \bigcup_{j=1}^{k+1} \{A_{i+1,j} \cup \{i+1\}\}.$$

Note that, F_{i+1} is starlike of size $k+1$ with center $i+1$, and $F_{i+1} \cap F_j = \emptyset$ for each j, $2 \le j \le i$. In a similar way, we can construct $F_{i+2}, \ldots, F_{k^2}$. If $n > k^2$, then for each l, $k^2+1 \le l \le n$, we define

$$F_l := \bigcup_{j=1}^{k+1} \{A_{2,j} \cup \{l\}\}.$$

From the above construction, it can be seen that, each F_j is starlike of size $k+1$ with center j. Also, every two vertices in F_j have j as their unique common symbol and hence, F_j uses exactly $1+(k+1)(k-1)$ symbols of the set $[n]$. Now, if $A \in \binom{[n]}{k} \setminus F_j$ is a vertex which $j \notin A$, then A has non-empty intersection with at most k elements of F_j for, A is a k-subset of $[n]$. Thus, there exists vertex $A' \in F_j$ such that $A \cap A' = \emptyset$ and hence, $d(A, F_j) = 1$. Therefore, properties (a) to (d) hold for this family.

Now, for each l, $2 \le l \le n$, we define the set $\bar{F}_l$ as

$$\bar{F}_l := \left\{ i_1 i_2 \ldots i_{k-1} l \mid i_1 i_2 \ldots i_{k-1} \in \binom{[l-1]}{k-1},\ i_1 i_2 \ldots i_{k-1} l \notin \bigcup_{j=2}^{n} F_j \right\}.$$

Thus, each non-empty $\bar{F}_l$ is starlike with center l. By letting $C_l := F_l \cup \bar{F}_l$, $2 \le l \le n$, it can be seen that $C_2, C_3, \ldots, C_n$ is a partition of the vertex set $\binom{[n]}{k}$ into starlike sets and hence we have a proper $(n-1)$-coloring of $KG(n,k)$.

Let $\prod = (C_2, \ldots, C_n)$ be an ordered partition of the vertex set into the above resulting color classes. Let $i_1 i_2 \ldots i_{k-1} l$ and $i'_1 i'_2 \ldots i'_{k-1} l$ be two distinct vertices in C_l. Since these two vertices are distinct, we can assume that

$$i_1 \notin \{i'_1, i'_2, \ldots, i'_{k-1}\}, \qquad i'_1 \notin \{i_1, i_2, \ldots, i_{k-1}\}.$$

Table 1
A locating 6-coloring of $KG(7,3)$.

C_1	C_2	C_3	C_4	C_5	C_6
123 (0, 2, 3, 1, 1, 1)	234 (1, 0, 2, 2, 2, 1)	345 (1, 1, 0, 2, 1, 3)	456 (1, 1, 2, 0, 2, 2)	467 (1, 2, 2, 1, 0, 2)	567 (1, 1, 2, 2, 2, 0)
124 (0, 2, 1, 2, 2, 1)	237 (1, 0, 2, 1, 1, 3)	346 (1, 1, 0, 1, 2, 3)	457 (1, 2, 2, 0, 1, 2)	126 (2, 2, 1, 1, 0, 3)	
127 (0, 2, 1, 1, 2, 3)	247 (1, 0, 1, 1, 1, 3)	347 (1, 1, 0, 1, 1, 3)	125 (2, 2, 1, 0, 1, 3)	136 (2, 1, 2, 1, 0, 3)	
134 (0, 1, 2, 2, 2, 1)	256 (1, 0, 1, 2, 2, 2)	356 (1, 1, 0, 2, 2, 2)	135 (2, 1, 2, 0, 1, 3)	146 (2, 1, 1, 1, 0, 3)	
137 (0, 1, 2, 1, 1, 3)	257 (1, 0, 1, 2, 1, 2)	357 (1, 2, 0, 2, 1, 2)	145 (2, 1, 1, 0, 1, 3)	236 (1, 2, 2, 1, 0, 3)	
147 (0, 1, 1, 1, 1, 3)	267 (1, 0, 1, 1, 2, 2)	367 (1, 2, 0, 1, 2, 2)	235 (1, 2, 2, 0, 1, 3)	246 (1, 2, 1, 1, 0, 3)	
156 (0, 1, 1, 2, 2, 2)			245 (1, 2, 1, 0, 1, 3)		
157 (0, 1, 1, 2, 1, 2)					
167 (0, 1, 1, 1, 2, 2)					

Without loss of generality, assume that $i'_1 < i_1$. By the properties of the family $\{F_2, F_3, \ldots, F_n\}$, it can be seen that

$$d(i_1 i_2 \ldots i_{k-1} l, C_{i_1}) = 2, \qquad d(i'_1 i'_2 \ldots i'_{k-1} l, C_{i_1}) = 1.$$

Hence, $c_\Pi(i_1 i_2 \ldots i_{k-1} l) \neq c_\Pi(i'_1 i'_2 \ldots i'_{k-1} l)$; accordingly, this coloring is a locating $(n-1)$-coloring. $\square$

The following result gives almost tight and good upper and lower bounds for the locating chromatic number of $KG(n,3)$, $n \geq 7$.

Theorem 4. *For all positive integers $n \geq 7$, we have*

$$n - 4 \leq \chi_L(KG(n,3)) \leq n - 1.$$

Proof. Since the chromatic number of the Kneser graph $KG(n,3)$ is $n-4$, the lower bound follows. By Theorem 3, for $n \geq 9$, we have $\chi_L(KG(n,3)) \leq n-1$. For the cases $n = 7$ and $n = 8$, we have explicit locating $(n-1)$-colorings. The color classes and the color codes of the vertices are illustrated in Tables 1 and 2. $\square$

Theorems 2–4 motivate us to propose the following question.

Question. *Is it true that for all positive integers $n, k,\ n > 2k$, $\chi_L(KG(n,k)) = n - 1$?*

4. The locating chromatic number of $KG(2k+1,k)$

Let $k \geq 3$. Note that, $\chi(KG(2k+1,k)) = 3$ and by Theorem C, the diameter of $KG(2k+1,k)$ is k.

In what follows, we present a lower bound for the locating chromatic number of odd graphs, which shows that the chromatic number and the locating chromatic number of odd graphs are far apart. Moreover, we give an upper bound for the metric dimension and accordingly for the locating chromatic number of odd graphs.

Lemma 1. *If $k \geq 4$, then $\binom{2k+1}{k} \leq k^k$.*

Proof. By a simple calculation, it can be seen that the lemma holds for $k \in \{4,5,6\}$. Obviously, for all $k \geq 7$, we have $k^2 + 5k + 6 < 2k^2$. Thus, $\frac{k+3}{2k} \times \frac{k+2}{k} < 1$.

Moreover, for each j, $4 \leq j \leq k+1$, we have $\frac{k+j}{k(j-1)} < 1$. Therefore,

$$\begin{aligned}\frac{\binom{2k+1}{k}}{k^k} &= \frac{2k+1}{k^2} \times \frac{2k}{k^2-k} \times \cdots \times \frac{k+3}{2k} \times \frac{k+2}{k} \\ &= \left(\prod_{j=4}^{k+1} \frac{k+j}{k(j-1)}\right) \times \frac{k+3}{2k} \times \frac{k+2}{k} < 1. \quad \square\end{aligned}$$

Theorem 5. *For all positive integers k, $k \geq 3$, we have* $\dim_M(KG(2k+1,k)) \leq 2k+1$, *and*

$$\left\lceil \frac{2k\ln 2 - \frac{1}{8k} - \frac{1}{2}\ln(k\pi)}{\ln k} \right\rceil \leq \chi_L(KG(2k+1,k)) \leq 2k+4.$$

Particularly, $\chi(KG(2k+1,k)) < \chi_L(KG(2k+1,k))$.

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Table 2
A locating 7-coloring of $KG(8, 3)$.

C_1	C_2	C_3	C_4	C_5	C_6	C_7
128 (0, 2, 1, 1, 1, 1, 1)	234 (1, 0, 2, 2, 1, 1, 1)	123 (2, 2, 0, 1, 1, 1, 1)	124 (2, 2, 1, 0, 1, 1, 1)	125 (2, 2, 1, 1, 0, 1, 1)	126 (2, 2, 1, 1, 1, 0, 1)	127 (2, 2, 1, 1, 1, 1, 0)
138 (0, 1, 2, 1, 1, 2, 2)	235 (1, 0, 2, 1, 2, 1, 1)	236 (1, 2, 0, 1, 1, 2, 1)	134 (2, 1, 2, 0, 1, 1, 1)	135 (2, 1, 2, 1, 0, 1, 1)	136 (2, 1, 2, 1, 1, 0, 1)	137 (2, 1, 2, 1, 1, 1, 0)
145 (0, 1, 1, 2, 2, 1, 1)	238 (1, 0, 2, 1, 1, 1, 1)	237 (1, 2, 0, 1, 1, 1, 2)	148 (2, 1, 1, 0, 1, 2, 2)	157 (2, 1, 1, 1, 0, 1, 2)	146 (2, 1, 1, 2, 1, 0, 1)	147 (2, 1, 1, 2, 2, 1, 0)
156 (0, 1, 1, 1, 2, 2, 1)	245 (1, 0, 1, 2, 2, 1, 1)	345 (1, 1, 0, 2, 2, 1, 1)	246 (1, 2, 1, 0, 1, 2, 1)	158 (2, 1, 1, 1, 0, 1, 1)	167 (2, 1, 1, 1, 1, 0, 2)	278 (1, 2, 1, 1, 1, 1, 0)
168 (0, 1, 1, 1, 1, 2, 1)	247 (1, 0, 1, 2, 1, 1, 2)	348 (1, 1, 0, 2, 1, 1, 1)	248 (1, 2, 1, 0, 1, 1, 1)	257 (1, 2, 1, 1, 0, 1, 2)	268 (1, 2, 1, 1, 1, 0, 1)	347 (1, 1, 2, 2, 1, 1, 0)
178 (0, 1, 1, 1, 2, 1, 2)	256 (1, 0, 1, 1, 2, 2, 1)	356 (1, 1, 0, 1, 2, 2, 1)	456 (1, 1, 1, 0, 2, 2, 1)	357 (1, 2, 2, 1, 0, 1, 2)	346 (1, 1, 2, 2, 1, 0, 1)	378 (1, 1, 2, 1, 1, 1, 0)
	258 (2, 0, 1, 1, 2, 1, 1)	358 (2, 1, 0, 1, 2, 1, 1)	457 (1, 1, 1, 0, 2, 1, 2)	458 (2, 1, 1, 2, 0, 1, 1)	568 (2, 1, 1, 1, 2, 0, 1)	478 (1, 1, 1, 2, 1, 1, 0)
	267 (1, 0, 1, 1, 1, 2, 2)	367 (1, 1, 0, 1, 1, 2, 2)	467 (1, 1, 1, 0, 1, 2, 2)	567 (1, 1, 1, 1, 0, 2, 2)	678 (1, 1, 1, 1, 1, 0, 2)	578 (2, 1, 1, 1, 2, 1, 0)
		368 (1, 1, 0, 1, 1, 2, 1)	468 (2, 1, 1, 0, 1, 2, 1)			

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Proof. By Theorem C, if A, B are two distinct vertices in $KG(2k+1,k)$, then

$$d(A,B) = \min\{2(k - |A \cap B|), 2|A \cap B| + 1\}.$$

This implies that the possible values of $d(A,B)$ are in one-to-one correspondence with the possible values $\{0, 1, 2, \ldots, k-1\}$ of $|A \cap B|$. Thus, we have $d(A,B) = d(A_1,B_1)$ if and only if $|A \cap B| = |A_1 \cap B_1|$, where $A_1, B_1 \in V(KG(2k+1,k))$. For each i, $1 \leq i \leq 2k+1$, let $w_i := \{i, i+1, \ldots, i+(k-1)\}$, where the elements are considered modulo $2k+1$. For example, $w_1 = \{1, 2, \ldots, k\}$ and $w_{2k+1} = \{2k+1, 1, 2, \ldots, k-1\}$. We want to show that for each two distinct vertices A, B, there exists a vertex w_i, $1 \leq i \leq 2k+1$, such that $d(A, w_i) \neq d(B, w_i)$, equivalently $|A \cap w_i| \neq |B \cap w_i|$. Suppose that, on the contrary, there exist vertices A and B such that $|A \cap w_i| = |B \cap w_i|$ for all $i \in \{1, 2, \ldots, 2k+1\}$. Let $A' := A \setminus B$ and $B' := B \setminus A$. Thus, we have

$$1 \leq |A'| = |B'| = k - |A \cap B| \leq k,$$

and $|A' \cap w_i| = |B' \cap w_i|$ for each $i \in \{1, 2, \ldots, 2k+1\}$. Let $a \in A'$. Since $A' \cap B' = \emptyset$, $a \notin B'$ and

$$|A' \cap w_a| = |B' \cap w_a| \leq |B' \cap w_{a+1}| = |A' \cap w_{a+1}|.$$

Moreover, $a \in A' \cap w_a$, $a \notin A' \cap w_{a+1}$ and $w_{a+1} \setminus w_a = \{a+k\}$. Therefore, $a + k \in A'$. Similarly, $a + jk \in A' \pmod{2k+1}$, for each $j \in \mathbb{N}$. If $a + jk \equiv a + j'k \pmod{2k+1}$, then $j \equiv j' \pmod{2k+1}$. This implies that $|A'| = 2k+1$, which is a contradiction. Hence, the set $\{w_1, \ldots, w_{2k+1}\}$ is a resolving set, and $\dim_M(KG(2k+1,k)) \leq 2k+1$. By Theorem B, we have

$$\chi_L(KG(2k+1,k)) \leq \dim_M(KG(2k+1,k)) + \chi(KG(2k+1,k)) \leq 2k+4.$$

Finally, we show that the lower bound holds. Using Theorem 1, it is easy to see that the inequality holds for $k = 3$. Suppose that $k \geq 4$, and let

$$l_0 := \min\{l \mid |V(KG(2k+1,k))| \leq l k^{l-1}\}.$$

By Theorem 1 and Lemma 1, we have $l_0 \leq \min\{\chi_L(KG(2k+1,k)), k\}$. Now, since

$$\frac{4^k e^{\frac{-1}{8k}}}{\sqrt{k\pi}} \leq \binom{2k}{k} < \binom{2k+1}{k} \leq l_0 k^{l_0 - 1} \leq k^{l_0},$$

where the first inequality is taken from [15], the lower bound follows. Note that, the lower bound is a non-decreasing function and for $k \geq 4$ is greater than three. Also, by Theorem 1, $\chi_L(KG(7,3)) \geq 4$. Therefore, $\chi(KG(n,k)) < \chi_L(KG(n,k))$. □

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