The Metric Dimension of Lexicographic Product of Graphs

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Abstract

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension. In this paper, we study the metric dimension of the lexicographic product of graphs G and H, $G[H]$. First, we introduce a new parameter which is called adjacency metric dimension of a graph. Then, we obtain the metric dimension of $G[H]$ in terms of the order of G and the adjacency metric dimension of H.

Keywords: Lexicographic product; Resolving set; Metric dimension; Basis; Adjacency metric dimension.

1 Introduction

In this section, we present some definitions and known results which are necessary to prove our main theorems. Throughout this paper, $G = (V, E)$ is a finite simple graph. We use $\overline{G}$ for the complement of graph G. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. Also, $N_G(v)$ is the set of all neighbors of vertex v in G. We write these simply $d(u, v)$ and $N(v)$, when no confusion can arise. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and none-adjacency relation between u and v, respectively. The symbols $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$ represent a path of order n, P_n, and a cycle of order n, C_n, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the (*metric*) *representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. In this case, we say set W resolves G. Elements in a resolving set are called *landmarks*. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$. The concept of (metric) representation is introduced by Slater [11] (see [8]). For more results related to these concepts see [1, 5, 7, 12].

We say an ordered set W *resolves* a set T of vertices in G, if the representations of vertices in T are distinct with respect to W. When $W = \{x\}$, we say that vertex x resolves T. To see that whether a given set W is a resolving set for G, it is sufficient to look at the representations of vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$.

Two distinct vertices u, v are said *twins* if $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. It is called that $u \equiv v$ if and only if $u = v$ or u, v are twins. In [9], it is proved that "$\equiv$" is an equivalent relation. The equivalence class of vertex v is denoted by v^*. Hernando et al. [9] proved that v^* is a clique or an independent set in G. As in [9], we say v^* is of type (1), (K), or (N) if v^* is a class of size 1, a clique of size at least 2, or an independent set of size at least 2. We denote the number of equivalence classes of G with respect to "$\equiv$" by $\iota(G)$. We mean by $\iota_K(G)$ and $\iota_N(G)$, the number of classes of type (K) and type (N) in G, respectively. We also use $a(G)$ and $b(G)$ for the number of all vertices in G which have at least an adjacent twin and a none-adjacent twin vertex in G, respectively. On the other way, $a(G)$ is the number of all vertices in the classes of type (K) and $b(G)$ is the number of all vertices in the classes of type (N). Clearly, $\iota(G) = n(G) - a(G) - b(G) + \iota_N(G) + \iota_K(G)$.

Observation 1. [9] *Suppose that u, v are twins in a graph G and W resolves G. Then u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves G.*

Theorem A. [6] Let G be a connected graph of order n. Then,

(i) $\beta(G) = 1$ if and only if $G = P_n$,

(ii) $\beta(G) = n - 1$ if and only if $G = K_n$.

Theorem B. [2, 3]

(i) If $n \notin \{3, 6\}$, then $\beta(C_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$,

(ii) If $n \notin \{1, 2, 3, 6\}$, then $\beta(P_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

The metric dimension of cartesian product of graphs is studied by Caseres et al. in [4]. They obtained the metric dimension of cartesian product of graphs G and H, $G\Box H$, where $G, H \in \{P_n, C_n, K_n\}$.

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is a graph with vertex set $V(G) \times V(H) := \{(v, u) \mid v \in V(G), u \in V(H)\}$, where two vertices (v, u) and (v', u') are adjacent whenever, $v \sim v'$, or $v = v'$ and $u \sim u'$. When the order of G is at least 2, it is easy to see that $G[H]$ is a connected graph if and only if G is a connected graph.

This paper is aimed to investigate the metric dimension of lexicographic product of graphs. The main goal of Section 2 is introducing a new parameter, which we call it adjacency metric dimension. In Section 3, we prove some relations to determine the metric dimension of lexicographic product of graphs, $G[H]$, in terms of the order of G and the adjacency metric dimension of H. As a corollary of our main theorems, we obtain the exact value of the metric dimension of $G[H]$, where $G = C_n (n \geq 5)$ or $G = P_n (n \geq 4)$, and $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_{m_1,\ldots,m_t}, \overline{K}_{m_1,\ldots,m_t}\}$.

2 Adjacency Resolving Sets

S. Khuller et al. [10] have considered the application of the metric dimension of a connected graph in robot navigation. In that sense, a robot moves from node to node of a graph space. If the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggest the problem of finding the fewest number of landmarks needed, and where should be located, so that the distances to the landmarks uniquely determine the robot's position on the graph. The solution of this problem is the metric dimension and a basis of the graph.

Now let there exist a large number of landmarks, but the cost of computing distance is much for the robot. In this case, robot can determine its position on the graph only by knowing landmarks which are adjacent to it. Here, the problem of finding the fewest number of landmarks needed, and where should be located, so that the adjacency and none-adjacency to the landmarks uniquely determine the robot's position on the graph is a different problem. The answer to this problem is one of the motivations of introducing *adjacency resolving sets* in graphs.

Definition 1. *Let G be a graph and $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. For each vertex $v \in V(G)$ the adjacency representation of v with respect to W is k-vector*

$$r_2(v|W) := (a_G(v, w_1), a_G(v, w_2), \ldots, a_G(v, w_k)),$$

where

$$a_G(v, w_i) = \begin{cases} 0 & \text{if } v = w_i, \\ 1 & \text{if } v \sim w_i, \\ 2 & \text{if } v \nsim w_i. \end{cases}$$

If all distinct vertices of G have distinct adjacency representations, W is called an adjacency resolving set for G. The minimum cardinality of an adjacency resolving set is called adjacency metric dimension of G, denoted by $\beta_2(G)$. An adjacency resolving set of cardinality $\beta_2(G)$ is called an adjacency basis of G.

By the definition, if G is a connected graph with diameter 2, then $\beta(G) = \beta_2(G)$. The converse is false; it can be seen that $\beta_2(C_6) = 2 = \beta(C_6)$ while, $diam(C_6) = 3$.

In the following, we obtain some useful results on the adjacency metric dimension of graphs.

Proposition 1. *For every connected graph G, $\beta(G) \leq \beta_2(G)$.*

Proof. Let W be an adjacency basis of G. Thus, for each pair of vertices $u, v \in V(G)$ there exist a vertex $w \in W$ such that, $a_G(u, w) \neq a_G(v, w)$. Therefore, $d_G(u, w) \neq d_G(v, w)$ and hence W is a resolving set for G. ■

Proposition 2. *For every graph G, $\beta_2(G) = \beta_2(\overline{G})$.*

Proof. Let W be an adjacency basis of G. For each pair of vertices $u, v \in V(G)$, there exist a vertex $w \in W$ such that $a_G(u, w) \neq a_G(v, w)$. Without loss of generality, assume that $a_G(u, w) < a_G(v, w)$. Thus, if $a_G(u, w) = 0$, then $a_{\overline{G}}(u, w) = 0$ and $a_{\overline{G}}(v, w) > 0$. Also, if $a_G(u, w) = 1$, then $a_G(v, w) = 2$ and hence, $a_{\overline{G}}(u, w) = 2$ and $a_{\overline{G}}(v, w) = 1$. Therefore, W is an adjacency resolving set for $\overline{G}$ and $\beta_2(\overline{G}) \leq \beta_2(G)$. Since $\overline{\overline{G}} = G$, we conclude that $\beta_2(G) \leq \beta_2(\overline{G})$ and consequently, $\beta_2(G) = \beta_2(\overline{G})$. ■

Let G be a graph of order n. It is easy to see that, $1 \le \beta_2(G) \le n-1$. In the following proposition, we characterize all graphs G with $\beta_2(G)=1$ and all graphs G of order n and $\beta_2(G)=n-1$.

Proposition 3. *If G is a graph of order n, then*

(i) $\beta_2(G)=1$ if and only if $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$.

(ii) $\beta_2(G)=n-1$ if and only if $G=K_n$ or $G=\overline{K}_n$.

Proof. (i) It is easy to see that for $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$, $\beta_2(G)=1$. Conversely, let G be a graph with $\beta_2(G)=1$. If G is a connected graph, then by Proposition 1, $\beta(G) \le \beta_2(G)=1$. Thus, by Theorem A, $G=P_n$. If $n \ge 4$, then $\beta_2(P_n) \ge 2$. Hence, $n \le 3$. If G is a disconnected graph and $\beta_2(G)=1$, then $\overline{G}$ is a connected graph and by Proposition 2, $\beta_2(\overline{G})=1$. Thus, $\overline{G}=P_n$, $n \in \{2,3\}$. Therefore, $G=\overline{P}_2$ or $G=\overline{P}_3$.

(ii) By Proposition 1, we have $n-1=\beta(K_n) \le \beta_2(K_n)$. On the other hand, $\beta_2(G) \le n-1$. Therefore, $\beta_2(K_n)=n-1$ and by Proposition 2, $\beta_2(\overline{K}_n)=\beta_2(K_n)=n-1$. Conversely, let G be a connected graph with $\beta_2(G)=n-1$. Suppose on the contrary that $G \ne K_n$. Thus, P_3 is an induced subgraph of G. Let $P_3=(x_1,x_2,x_3)$. Therefore, $a_G(x_2,x_1)=1$ and $a_G(x_3,x_1)=2$. Consequently, $V(G)\backslash\{x_2,x_3\}$ is an adjacency resolving set for G of cardinality $n-2$. That is, $\beta_2(G) \le n-2$, which is a contradiction. Hence, $G=K_n$. If G is a disconnected graph with $\beta_2(G)=n-1$, then $\overline{G}$ is a connected graph and by Proposition 2, $\beta_2(\overline{G})=n-1$. Thus, $\overline{G}=K_n$. ■

Lemma 1. *If u is a vertex of degree $n(G)-1$ in a connected graph G, then G has a basis which does not include u.*

Proof. Let B be a basis of G which contains u. Thus, $r(u|B\backslash\{u\})=(1,\ldots,1)$. Since B is a basis of G, there exist two vertices $v,w \in V(G)\backslash(B\backslash\{u\})$ such that, $r(v|B\backslash\{u\})=r(w|B\backslash\{u\})$ and $d_G(u,v) \ne d_G(u,w)$. If $u \notin \{v,w\}$, then $d(u,v)=d(u,w)=1$, which is a contradiction. Hence, $u \in \{v,w\}$, say $u=v$. Therefore, $r(w|B\backslash\{u\})=r(u|B\backslash\{u\})=(1,1,\ldots,1)$ and for each $x,y \in V(G)\backslash\{u,w\}$, $r(x|B\backslash\{u\}) \ne r(y|B\backslash\{u\})$. Note that, $r(w|B)=(1,1,\ldots,1)$, because $u \sim w$. Since B is a basis of G, w is the unique vertex of G which its representation with respect to B is entirely 1. It implies that w is the unique vertex of $G\backslash B$ with $r(w|B\backslash\{u\})=(1,1,\ldots,1)$. Therefore, the set $(B\backslash\{u\})\cup\{w\}$ is a basis of G which does not contain u. ■

Proposition 4. *For every graph G, $\beta(G\vee K_1)-1 \le \beta_2(G) \le \beta(G\vee K_1)$. Moreover, $\beta_2(G)=\beta(G\vee K_1)$ if and only if G has an adjacency basis for which no vertex has adjacency representation entirely 1 with respect to it.*

Proof. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ and $V(K_1)=\{u\}$. Note that, $d_{G\vee K_1}(v_i,v_j)=a_G(v_i,v_j)$, $1 \le i,j \le n$. By Lemma 1, $G \vee K_1$ has a basis $B=\{b_1,b_2,\ldots,b_k\}$ such that $u \notin B$. Therefore,

$$r(v_i|B)=(d_{G\vee K_1}(v_i,b_1), d_{G\vee K_1}(v_i,b_2),\ldots,d_{G\vee K_1}(v_i,b_k))=r_2(v_i|B)$$

for each v_i, $1 \le i \le n$. Thus, B is an adjacency resolving set for G and $\beta_2(G) \le \beta(G\vee K_1)$.

Now let $W=\{w_1,w_2,\ldots,w_t\}$ be an adjacency basis of G. Since $d_{G\vee K_1}(v_i,w_j)=a_G(v_i,w_j)$, $1 \le i \le n$, $1 \le j \le t$, we have $r(v_i|W)=r_2(v_i|W)$, $1 \le i \le n$. Hence, W resolves $V(G\vee K_1)\backslash\{u\}$ and $\beta(G\vee K_1)-1 \le \beta_2(G)$. On the other hand, $r(u|W)$ is entirely 1. Therefore, W is a resolving set for $G\vee K_1$ if and only if $r_2(v_i|W)$ is not entirely 1 for every v_i, $1 \le i \le n$. Since $\beta_2(G) \le \beta(G\vee K_1)$, we have $\beta_2(G)=\beta(G\vee K_1)$ if and only if $r_2(v_i|W)$ is not entirely 1 for every v_i, $1 \le i \le n$. ■

Proposition 5. *If $n \geq 4$, then $\beta_2(C_n) = \beta_2(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.*

Proof. If $n \leq 8$, then by a simple computation, we can see that $\beta_2(C_n) = \beta_2(P_n) = \lfloor \frac{2n+2}{5} \rfloor$. Now, let $G \in \{P_n, C_n\}$, and $n \geq 9$. By Theorem B, $\beta(G \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor \geq 4$. Hence, by Proposition 4, we have $\beta_2(G) \geq 3$. If W is an adjacency basis of G, then for each vertex $v \in V(G)$, $r_2(v|W)$ is not entirely 1, because v has at most two neighbors. Therefore, by Proposition 4, $\beta_2(G) = \beta_2(G \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$. ■

Proposition 6. *If $K_{m_1,m_2,\ldots,m_t}$ is the complete t-partite graph, then*

$$\beta_2(K_{m_1,m_2,\ldots,m_t}) = \beta(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} m - r - 1 & \text{if } r \neq t, \\ m - r & \text{if } r = t, \end{cases}$$

where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$.

Proof. Since $diam(K_{m_1,m_2,\ldots,m_t}) = 2$, we have $\beta_2(K_{m_1,m_2,\ldots,m_t}) = \beta(K_{m_1,m_2,\ldots,m_t})$. Let M_i be the partite set of size m_i, $1 \leq i \leq t$. For each i, $1 \leq i \leq r$, all vertices of M_i are none-adjacent twins. Also, all vertices of $\cup_{i=r+1}^{t} M_i$ are adjacent twins. Let x_i be a fixed vertex in M_i, $1 \leq i \leq r$. If $r = t$, then by Observation 1, $\beta(K_{m_1,m_2,\ldots,m_t}) \geq \sum_{i=1}^{t} m_i - r$. Also, the set $\cup_{i=1}^{t} M_i \backslash \{x_1, x_2, \ldots, x_r\}$ is a resolving set for $K_{m_1,m_2,\ldots,m_t}$ with cardinality $\sum_{i=1}^{t} m_i - r$. Thus, $\beta(K_{m_1,m_2,\ldots,m_t}) = \sum_{i=1}^{t} m_i - r = m - r$. If $r \neq t$, then $\cup_{i=r+1}^{t} M_i \neq \emptyset$. Let $x_{r+1} \in \cup_{i=r+1}^{t} M_i$. Observation 1 implies that $\beta(K_{m_1,m_2,\ldots,m_t}) \geq \sum_{i=1}^{t} m_i - r - 1$. On the other hand, the set $\cup_{i=1}^{t} M_i \backslash \{x_1, x_2, \ldots, x_{r+1}\}$ is a resolving set for $K_{m_1,m_2,\ldots,m_t}$ with cardinality $\sum_{i=1}^{t} m_i - r - 1 = m - r - 1$. ■

3 Lexicographic Product of Graphs

Throughout this section, G is a connected graph of order n, $V(G) = \{v_1, v_2, \ldots, v_n\}$, H is a graph of order m, and $V(H) = \{u_1, u_2, \ldots, u_m\}$. Therefore, $G[H]$ is a connected graph. For convenience, we denote the vertex (v_i, u_j) of $G[H]$ by v_{ij}. Note that, for each pair of vertices $v_{ij}, v_{rs} \in V(G[H])$,

$$d_{G[H]}(v_{ij}, v_{rs}) = \begin{cases} d_G(v_i, v_r) & \text{if } v_i \neq v_r, \\ 1 & \text{if } v_i = v_r \text{ and } u_j \sim u_s, \\ 2 & \text{if } v_i = v_r \text{ and } u_j \nsim u_s. \end{cases}$$

On the other words,

$$d_{G[H]}(v_{ij}, v_{rs}) = \begin{cases} d_G(v_i, v_r) & \text{if } v_i \neq v_r, \\ a_H(u_j, u_s) & \text{otherwise.} \end{cases}$$

Let S be a subset of $V(G[H])$. The *projection* of S onto H is the set $\{u_j \in V(H) \mid v_{ij} \in S\}$. Also, the ith *row* of $G[H]$, denoted by H_i, is the set $\{v_{ij} \in V(G[H]) \mid 1 \leq j \leq m\}$.

Lemma 2. *If $W \subseteq V(G[H])$ is a resolving set for $G[H]$, then $W \cap H_i$ resolves H_i, for each i, $1 \leq i \leq n$. Moreover, the projection of $W \cap H_i$ onto H is an adjacency resolving set for H, for each i, $1 \leq i \leq n$.*

Proof. Since W resolves $G[H]$, for each pair of vertices $v_{ij}, v_{iq} \in H_i$, there exist a vertex $v_{rt} \in W$ such that, $d_{G[H]}(v_{rt}, v_{ij}) \neq d_{G[H]}(v_{rt}, v_{iq})$. If $r \neq i$, then $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) = d_{G[H]}(v_{rt}, v_{iq})$, which is a contradiction. Therefore, $i = r$ and $W \cap H_i$ resolves H_i.

Now, let $u_j, u_q \in V(H)$. Since $W \cap H_i$ resolves H_i, there exist a vertex $v_{it} \in W \cap H_i$ such that, $d_{G[H]}(v_{it}, v_{ij}) \neq d_{G[H]}(v_{it}, v_{iq})$. Hence, $a_H(u_t, u_j) = d_{G[H]}(v_{it}, v_{ij}) \neq d_{G[H]}(v_{it}, v_{iq}) = a_H(u_t, u_q)$. Consequently, the projection of $W \cap H_i$ onto H is an adjacency resolving set for H. ■

By Lemma 2, every basis of $G[H]$ contains at least $\beta_2(H)$ vertices from each copy of H in $G[H]$. Thus, the following lower bound for $\beta(G[H])$ is obtained.

$$\beta(G[H]) \geq n\beta_2(H). \tag{1}$$

Theorem 1. *Let G be a connected graph of order n and H be an arbitrary graph. If there exist two adjacency bases W_1 and W_2 of H such that, there is no vertex with adjacency representation entirely 1 with respect to W_1 and no vertex with adjacency representation entirely 2 with respect to W_2, then $\beta(G[H]) = \beta(G[\overline{H}]) = n\beta_2(H)$.*

Proof. By Inequality 1, we have $\beta(G[H]) \geq n\beta_2(H)$. To prove the equality, it is enough to provide a resolving set for $G[H]$ of size $n\beta_2(H)$. For this sake, let

$$S = \{v_{ij} \in V(G[H]) \mid v_i \in K(G),\ u_j \in W_1\} \cup \{v_{ij} \in V(G[H]) \mid v_i \notin K(G),\ u_j \in W_2\},$$

where $K(G)$ is the set of all vertices of G in equivalence classes of type (K). On the other word, $K(G)$ is the set of all vertices of G which have adjacent twins. We show that S is a resolving set for $G[H]$. Let $v_{rt}, v_{pq} \in V(G[H])\backslash S$ be two distinct vertices. The following possibilities can be happened.

1. $r = p$. Note that, $v_{rt} \neq v_{pq}$ implies that $t \neq q$. Since W_1 and W_2 are adjacency resolving sets, there exist vertices $u_j \in W_1$ and $u_l \in W_2$ such that, $a_H(u_t, u_j) \neq a_H(u_q, u_j)$ and $a_H(u_t, u_l) \neq a_H(u_q, u_l)$. If $v_r \in K(G)$, then $v_{rj} \in S$ and $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) \neq a_H(u_q, u_j) = d_{G[H]}(v_{pq}, v_{rj})$. Similarly, if $v_r \notin K(G)$, then $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

2. $r \neq p$ and $v_r, v_p \in K(G)$. If v_r and v_p are not twins, then there exist a vertex $v_i \in V(G)\backslash\{v_r, v_p\}$ which is adjacent to only one of the vertices v_r and v_p. Hence, for each $u_j \in W_1$, we have $v_{ij} \in S$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. If v_r and v_p are twins, then $v_r \sim v_p$, because $v_r, v_p \in K(G)$. Since $r_2(u_t|W_1)$ is not entirely 1, there exist a vertex $u_l \in W_1$ such that, $a_H(u_t, u_l) = 2$. Therefore, $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) = a_H(u_t, u_l) = 2$. On the other hand, $d_{G[H]}(v_{pq}, v_{rl}) = d_G(v_p, v_r) = 1$. Thus, $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

3. $r \neq p$, $v_r \in K(G)$, and $v_q \notin K(G)$. In this case, v_r and v_p are not twins. Therefore, there exist a vertex $v_i \in V(G)\backslash\{v_r, v_p\}$ which is adjacent to only one of the vertices v_r and v_p. Let u_j be a vertex of $W_1 \cup W_2$, such that $v_{ij} \in S$. Hence, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$.

4. $r \neq p$ and $v_r, v_p \notin K(G)$. If v_r and v_p are not twins, then there exist a vertex $v_i \in V(G)\backslash\{v_r, v_p\}$ which is adjacent to only one of the vertices v_r and v_p. Thus, for each $u_j \in W_2$, we have $v_{ij} \in S$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. If v_r and v_p are twins, then $v_r \nsim v_p$, because $v_r, v_p \notin K(G)$. Since $r_2(u_t|W_2)$ is not entirely 2, there exist a vertex $u_l \in W_2$, such that $a_H(u_t, u_l) = 1$. Therefore, $v_{rl} \in S$ and $d_{G[H]}(v_{rt}, v_{rl}) = a_H(u_t, u_l) = 1$. On the other hand, $d_{G[H]}(v_{pq}, v_{rl}) = d_G(v_p, v_r) = 2$, since v_r and v_p are none-adjacent twins in the connected G. Hence, $d_{G[H]}(v_{rt}, v_{rl}) \neq d_{G[H]}(v_{pq}, v_{rl})$.

Thus, $r(v_{rt}|S) \neq r(v_{pq}|S)$. Therefore, S is a resolving set for $G[H]$ with cardinality $n\beta_2(H)$.

Clearly, in $\overline{H}$, for each $u \in V(\overline{H})$, $r_2(u|W_1)$ is not entirely 2 and $r_2(u|W_2)$ is not entirely 1. Since $\beta_2(H) = \beta_2(\overline{H})$, by interchanging the roles of W_1 and W_2 for $\overline{H}$, we conclude $\beta(G[\overline{H}]) = n\beta_2(\overline{H}) = n\beta_2(H)$. ■

In the following three theorems, we obtain $\beta(G[H])$, when H does not satisfy the assumption of Theorem 1.

Theorem 2. *Let G be a connected graph of order n and H be an arbitrary graph. If for each adjacency basis W of H there exist vertices with adjacency representations entirely 1 and entirely 2 with respect to W, then $\beta(G[H]) = \beta(G[\overline{H}]) = n(\beta_2(H)+1) - \iota(G)$.*

Proof. Let B be a basis of $G[H]$ and B_i be the projection of $B \cap H_i$ onto H, for each i, $1 \le i \le n$. By Lemma 2, B_i's are adjacency resolving sets for H. Therefore, $|B \cap H_i| = |B_i| \ge \beta_2(H)$ for each i, $1 \le i \le n$.

Let $I = \{i \mid |B_i| = \beta_2(H)\}$. We claim that $|I| \le \iota(G)$, otherwise by the pigeonhole principle, there exist a pair of twin vertices $v_r, v_p \in V(G)$ such that, $|B_r| = |B_p| = \beta_2(H)$. Since B_r and B_p are adjacency bases of H, by the assumption there are vertices u_t and u_q with adjacency representations entirely 1 with respect to B_r and B_p, respectively. Also, there are vertices u'_t and u'_q with adjacency representations entirely 2 with respect to B_r and B_p, respectively. Hence, for each $u \in B_r$ and $u' \in B_p$, we have $u_t \sim u$, $u_{t'} \nsim u$, $u_q \sim u'$, and $u_{q'} \nsim u'$. If $v_r \sim v_p$, then for each $v_{ij} \in B$ one of the following cases can be happened.

1. $i \notin \{r,p\}$. Since v_r and v_p are twins, we have $d_G(v_r, v_i) = d_G(v_p, v_i)$. On the other hand, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i)$. Thus, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

2. $i = p \neq r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$. Since $v_i = v_p \sim v_r$, we have $d_G(v_r, v_i) = 1$. On the other hand $u_j \in B_p$ and hence, $a_H(u_q, u_j) = 1$. Therefore, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

3. $i = r \neq p$. Similar to previous case, $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j) = 1$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i) = 1$. Consequently, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

4. $i = p = r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j)$. Since, $u_j \in B_p = B_r$, we have $a_H(u_q, u_j) = 1 = a_H(u_t, u_j)$. Thus, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Hence, $v_r \sim v_p$ implies that $r(v_{rt}|B) = r(v_{pq}|B)$, which is a contradiction. Therefore, $v_r \nsim v_p$. Since G is a connected graph, none-adjacent twin vertices v_r and v_p have at least one common neighbor and thus, $d_G(v_r, v_p) = 2$. Consequently, by a same method as the case $v_r \sim v_p$, we can see that $r(v_{rt'}|B) = r(v_{pq'}|B)$, which contradicts the assumption that B is a basis of $G[H]$. Hence $|I| \le \iota(G)$. On the other hand, every basis of $G[H]$ has at least $\beta_2(H)+1$ vertices in H_i, where $i \notin I$. Therefore,

$$\begin{aligned}\beta(G[H]) = |B| = |\cup_{i=1}^{n} (B \cap H_i)| &\ge |I|\beta_2(H) + (n - |I|)(\beta_2(H)+1)\\ &= n\beta_2(H) + n - |I|\\ &\ge n(\beta_2(H)+1) - \iota(G).\end{aligned}$$

Now let W be an adjacency basis of H. By assumption, there exist vertices $u_1, u_2 \in V(H)\backslash W$ such that, u_1 is adjacent to all vertices of W and u_2 is not adjacent to any vertex of W. Also, let $K(G)$ be the set of all classes of type (K), and $N(G)$ be the set of all classes of G of type (N) in G. Choose fixed vertex v from v^* for each $v^* \in N(G) \cup K(G)$. We claim that the set

$$S = \{v_{ij} \in V(G[H]) \mid u_j \in W\} \cup \{v_{t1} \mid v_t \in \cup_{v^* \in K(G)} (v^* \backslash \{v\})\} \cup \{v_{t2} \mid v_t \in \cup_{v^* \in N(G)} (v^* \backslash \{v\})\}$$

is a resolving set for $G[H]$. Let $v_{rt}, v_{pq} \in V(G[H])\backslash S$. Hence, one of the following cases can be happened.

1. $r = p$. Since W is an adjacency basis of H, there exist a vertex $u_j \in W$, such that $a_H(u_q, u_j) \neq a_H(u_t, u_j)$. Therefore, $d_{G[H]}(v_{pq}, v_{rj}) = a_H(u_q, u_j) \neq a_H(u_t, u_j) = d_{G[H]}(v_{rt}, v_{rj})$. Consequently, $r(v_{rt}|S) \neq r(v_{pq}|S)$.

2. $r \neq p$ and v_r, v_p are not twins. Hence, there exist a vertex $v_i \in V(G)$ which is adjacent to only one of the vertices v_r and v_p. Thus, for each vertex $u_j \in W$, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i) \neq d_G(v_p, v_i) = d_{G[H]}(v_{pq}, v_{ij})$. This yields, $r(v_{rt}|S) \neq r(v_{pq}|S)$.

3. v_r and v_p are adjacent twins. Therefore, at least one of the vertices v_{r1} and v_{p1}, say v_{r1} belongs to S. Since $v_{rt} \notin S$, we have $t \neq 1$. Hence, there exists a vertex $u_j \in S$, such that $a_H(u_t, u_j) = 2$, otherwise $t = 1$. Consequently, $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 2$. On the other hand, $d_{G[H]}(v_{pq}, v_{rj}) = d_G(v_p, v_r) = 1$, because $v_r \sim v_p$. This gives, $r(v_{rt}|S) \neq r(v_{pq}|S)$.

4. v_r and v_p are none-adjacent twins. In this case, at least one of the vertices v_{r2} and v_{p2}, say v_{r2} belongs to S. Hence, $t \neq 2$ and there exists a vertex $u_j \in W$, such that $a_H(u_t, u_j) = 1$, otherwise $t = 2$. Therefore, $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 1 \neq 2 = d_G(v_p, v_r) = d_{G[H]}(v_{pq}, v_{rj})$. Thus, $r(v_{rt}|S) \neq r(v_{pq}|S)$.

Consequently, S is a resolving set for $G[H]$ with cardinality

$$|S| = n\beta_2(H) + a(G) - \iota_K(G) + b(G) - \iota_N(G) = n(\beta_2(H) + 1) - \iota(G).$$

Since all adjacency bases of H and $\overline{H}$ are the same, $\overline{H}$ satisfies the condition of the theorem. Hence, $\beta(G[\overline{H}]) = n(\beta_2(H) + 1) - \iota(G)$ and the proof is completed. ■

Theorem 3. *Let G be a connected graph of order n and H be an arbitrary graph. If H has the following properties*

(i) for each adjacency basis of H there exist a vertex with adjacency representation entirely 1,

(ii) there exist an adjacency basis W of H such that there is no vertex with adjacency representation entirely 2 *with respect to W,*

then $\beta(G[H]) = n\beta_2(H) + a(G) - \iota_K(G)$.

Proof. Let B be a basis of $G[H]$ and B_i be the projection of $B \cap H_i$ onto H, for each i, $1 \leq i \leq n$. By Lemma 2, B_i's are adjacency resolving sets for H. Therefore, $|B \cap H_i| = |B_i| \geq \beta_2(H)$ for each i, $1 \leq i \leq n$.

Let $I = \{i \mid |B_i| = \beta_2(H)\}$. We claim that $|I| \leq n - a(G) + \iota_K(G)$, otherwise by the pigeonhole principle, there exist a pair of adjacent twin vertices $v_r, v_p \in V(G)$, such that $|B_r| = |B_p| = \beta_2(H)$. Since B_r and B_p are adjacency bases of H, by assumption (i) there exist vertices $u_t, u_q \in V(H)$ with adjacency representation entirely 1 with respect to B_r and B_p, respectively. Hence, for each $u \in B_r$ and each $u' \in B_p$, we have $u_t \sim u$, and $u_q \sim u'$. Since $v_r \sim v_p$, for each $v_{ij} \in B$ one of the following cases can be happened.

1. $i \notin \{r, p\}$. Since v_r and v_p are twins, we have $d_G(v_r, v_i) = d_G(v_p, v_i)$. On the other hand, $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i)$. Thus, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

2. $i = p \neq r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = d_G(v_r, v_i)$. Since

$v_i = v_p \sim v_r$, we have $d_G(v_r, v_i) = 1$. On the other hand $u_j \in B_p$ and hence, $a_H(u_q, u_j) = 1$. Therefore, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

3. $i = r \neq p$. Similar to previous case, $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j) = 1$ and $d_{G[H]}(v_{pq}, v_{ij}) = d_G(v_p, v_i) = 1$. Consequently, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

4. $i = p = r$. In this case, $d_{G[H]}(v_{pq}, v_{ij}) = a_H(u_q, u_j)$ and $d_{G[H]}(v_{rt}, v_{ij}) = a_H(u_t, u_j)$. Since, $u_j \in B_p = B_r$, we have $a_H(u_q, u_j) = 1 = a_H(u_t, u_j)$. Thus, $d_{G[H]}(v_{rt}, v_{ij}) = d_{G[H]}(v_{pq}, v_{ij})$.

Hence, $r(v_{rt}|B) = r(v_{pq}|B)$, which is a contradiction. Therefore, $|I| \leq n - a(G) + \iota_K(G)$. On the other hand, every basis of $G[H]$ has at least $\beta_2(H) + 1$ vertices in H_i, where $i \notin I$. Thus,

$$\begin{aligned}\beta(G[H]) = |B| &\geq |I|\beta_2(H) + (n - |I|)(\beta_2(H) + 1)\\ &= n\beta_2(H) + n - |I|\\ &\geq n\beta_2(H) + a(G) - \iota_K(G).\end{aligned}$$

Now let $K(G)$ be the set of all classes of type (K) in G and $v \in v^*$ be a fixed vertex for each class v^* of type (K). Also, let $u_1 \in V(H)\backslash W$, such that $r_2(u_1|W)$ is entirely 1. Consider

$$S = \{v_{ij} \in V(G[H]) \,|\, u_j \in W\} \cup \{v_{t1} \,|\, v_t \in \cup_{v^* \in K(G)}(v^* \backslash \{v\})\}$$

and let $v_{rt}, v_{pq} \in V(G[H])\backslash S$. If v_r and v_p are not none-adjacent twins, then similar to the proof of Theorem 2, we have $r(v_{rt}|S) \neq r(v_{pq}|S)$. Now, let v_r and v_p be none-adjacent twin vertices of G. By assumption, there exists a vertex $u_j \in W$, such that $a_H(u_t, u_j) = 1$. Therefore, $d_{G[H]}(v_{rt}, v_{rj}) = a_H(u_t, u_j) = 1$. On the other hand, $d_{G[H]}(v_{pq}, v_{rj}) = d_G(v_p, v_r) = 2$, since v_r and v_p are none-adjacent twins in the connected graph G. Hence, $r(v_{rt}|S) \neq r(v_{pq}|S)$. This implies that S is a resolving set for $G[H]$ with cardinality $n\beta_2(H) + a(G) - \iota_K(G)$. ■

By a similar proof, we have the following theorem.

Theorem 4. *Let G be a connected graph of order n and H be an arbitrary graph. If H has the following properties*

(i) for each adjacency basis of H there exist a vertex with adjacency representation entirely 2,

(ii) there exist an adjacency basis W of H such that there is no vertex with adjacency representation entirely 1 with respect to W,

then $\beta(G[H]) = n\beta_2(H) + b(G) - \iota_N(G)$.

Corollary 1. *If G has no pair of twin vertices, then $\beta(G[H]) = n\beta_2(H)$.*

Proof. The adjacency bases of H satisfy one of the conditions of Theorems 1, 2, 3, and 4 . Now, if G does not have any pair of twin vertices, then $\iota(G) = n$, $\iota_K(G) = a(G) = 0$, and $\iota_N(G) = b(G) = 0$. Therefore, $\beta(G[H]) = n\beta_2(H)$. ■

By Theorems 1, 2, 3, and 4 the exact value of $\beta(G[H])$ of many graphs G and H can be determined. In the following two corollaries, $\beta(G[H])$ for some of the well known graphs are obtained.

Corollary 2. *Let $G = P_n$, $n \geq 4$ or $G = C_n$, $n \geq 5$. Then, G does not have any pair of twin vertices. Thus by Corollary 1, $\beta(G[H]) = n\beta_2(H)$, for each graph H. In particular, by Propositions 2 and 5, $\beta_2(P_m) = \beta_2(C_m) = \beta_2(\overline{P}_m) = \beta_2(\overline{C}_m) = \lfloor \frac{2m+2}{5} \rfloor$. Therefore, $\beta(G[P_m]) = \beta(G[C_m]) = \beta(G[\overline{P}_m]) = \beta(G[\overline{C}_m]) = n\lfloor \frac{2m+2}{5} \rfloor$. Also, by Propositions 2 and 6, we have*

$$\beta(G[\overline{K}_{m_1,m_2,\ldots,m_t}]) = \beta(G[K_{m_1,m_2,\ldots,m_t}]) = \begin{cases} n(m-r-1) & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t, \end{cases}$$

where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$.

Corollary 3. *Let $H = K_{m_1,m_2,\ldots,m_t}$, where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$. Thus, for each adjacency basis of H there is no vertex of H with adjacency representation entirely 2.*

If $r = t$, then for each adjacency basis of H there is no vertex of H with adjacency representation entirely 1. Therefore, by Theorem 1, $\beta(G[H]) = n\beta_2(H)$ for each connected graph G of order n. If $r \neq t$, then for each adjacency basis of H, there exist a vertex with adjacency representation entirely 1. Thus, by Theorem 3, $\beta(G[H]) = n\beta_2(H) + a(G) - \iota_K(G)$ for each connected graph G of order n.

In particular, if $G = K_n$, then all vertices of K_n are adjacent twins. Thus, $a(K_n) = n$ and $\iota_K(K_n) = 1$, hence, $\beta(K_n[H]) = n\beta_2(H) + n - 1$. Therefore, by Proposition 6,

$$\beta(K_n[H]) = \begin{cases} n(m-r)-1 & \text{if } r \neq t, \\ n(m-r) & \text{if } r = t. \end{cases}$$

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